Birnbaum-Saunders Distribution

Debasis Kundu

Department of Mathematics & Statistics
Indian Institute of Technology Kanpur

30-th April, 2010

Outline

Outline

Normal Random Variable

- Z is called the standard normal random variable if it has the PDF;

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du.$$

Normal Random Variable

- Z is called the standard normal random variable if it has the PDF;

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du.$$

Normal Random Variable

- Z is called the standard normal random variable if it has the PDF;

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$\Phi(x) = \int_{-\infty}^{x} \phi(u) du.$$

Normal Random Variable

- If we make the transformation

$$X = \mu + \sigma Z$$

then X the PDF;

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$F(x) = \int_{-\infty}^{x} f(u; \mu, \sigma) du = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Normal Random Variable

- If we make the transformation

$$X = \mu + \sigma Z$$

then X the PDF;

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$F(x) = \int_{-\infty}^{x} f(u; \mu, \sigma) du = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Normal Random Variable

- If we make the transformation

$$X = \mu + \sigma Z$$

then X the PDF;

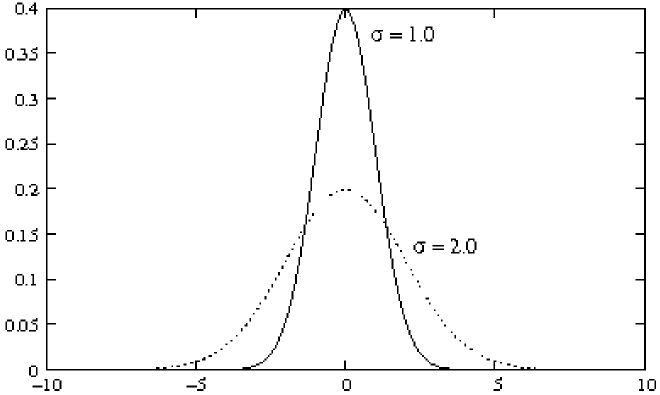
$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty.$$

- The corresponding CDF becomes;

$$F(x) = \int_{-\infty}^{x} f(u; \mu, \sigma) du = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

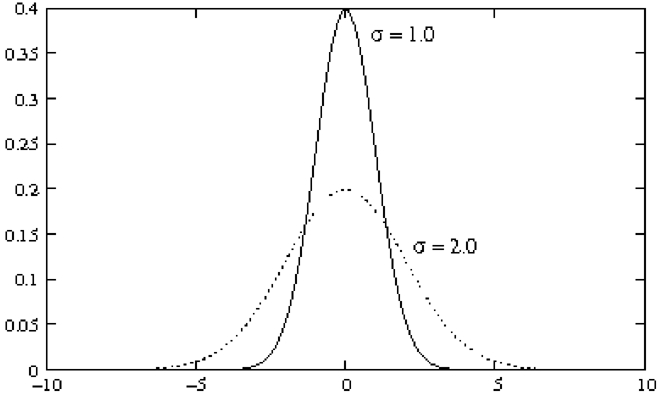
Normal Random Variable

- The shape of the normal PDF is symmetric about its mean μ.



Normal Random Variable

- The shape of the normal PDF is symmetric about its mean μ.



Outline

Transformation from Normal Random Variable

- Different random variables have been derived from normal random variables.

- Log-normal random variable: If X is a normal random variable then $Y = e^X$ has log-normal random variable.

$$F_Y(y) = \Phi(\ln y)$$

- The PDF of Log-normal random distribution is;

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

Transformation from Normal Random Variable

- Different random variables have been derived from normal random variables.

- Log-normal random variable: If X is a normal random variable then $Y = e^X$ has log-normal random variable.

$$F_Y(y) = \Phi(\ln y)$$

- The PDF of Log-normal random distribution is;

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$$

Transformation from Normal Random Variable

- Different random variables have been derived from normal random variables.
- Log-normal random variable: If X is a normal random variable then $Y = e^X$ has log-normal random variable.

$$F_Y(y) = \Phi(\ln y)$$

- The PDF of Log-normal random distribution is;

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

Transformation from Normal Random Variable

- Different random variables have been derived from normal random variables.
- Log-normal random variable: If X is a normal random variable then $Y = e^X$ has log-normal random variable.

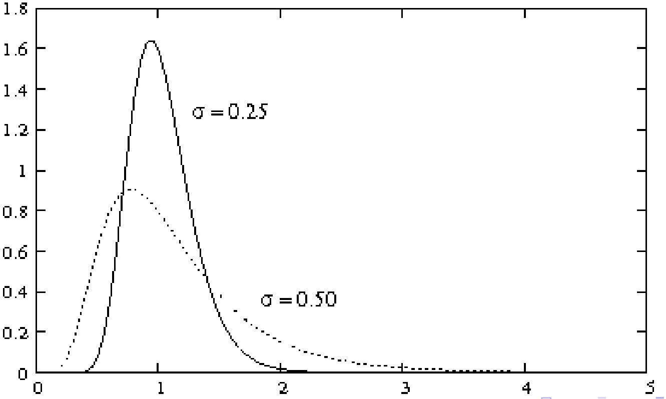
$$F_Y(y) = \Phi(\ln y)$$

- The PDF of Log-normal random distribution is;

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

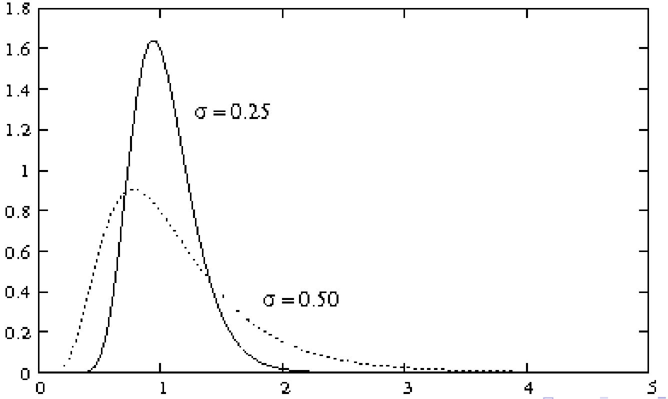
Log-normal Random Variable

- The shape of the log-normal PDF for different σ.



Log-normal Random Variable

- The shape of the log-normal PDF for different σ.



Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Log-normal distribution: Closeness

- The shape of the PDF of log-normal distribution is always unimodal.
- The PDF of the log-normal distribution is right skewed.
- The PDF of log-normal distribution has been used very successfully to model right skewed data.
- It is observed that the PDF of log-normal distribution is very similar to the shape of the PDF of well known Weibull distribution.
- Quite a bit of work has been done in discriminating between these two distribution functions.

Inverse Gaussian random variable

- Inverse Gaussian random variable has the following PDF:

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

- Inverse Gaussian has the CDF;

$$F(x;\mu,\sigma) = \Phi\left\{\frac{\lambda}{x}\left(\frac{x}{\mu}-1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\frac{\lambda}{x}\left(\frac{x}{\mu}+1\right)\right\}$$

Inverse Gaussian random variable

- Inverse Gaussian random variable has the following PDF:

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

- Inverse Gaussian has the CDF;

$$F(x; \mu, \sigma) = \Phi\left\{\frac{\lambda}{x}\left(\frac{x}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\frac{\lambda}{x}\left(\frac{x}{\mu} + 1\right)\right\}$$

Inverse Gaussian random variable

- Inverse Gaussian random variable has the following PDF:

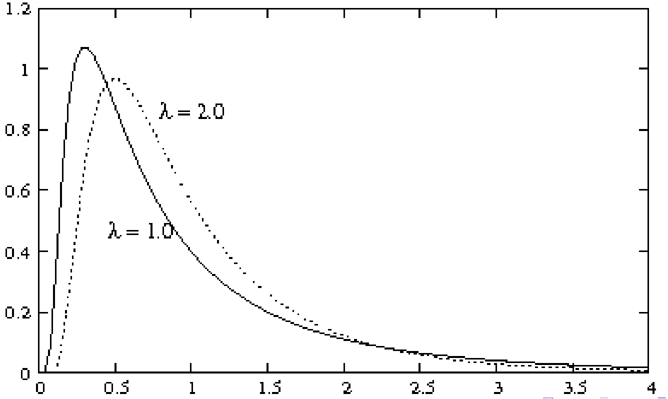
$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

- Inverse Gaussian has the CDF;

$$F(x;\mu,\sigma) = \Phi\left\{\frac{\lambda}{x}\left(\frac{x}{\mu}-1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\frac{\lambda}{x}\left(\frac{x}{\mu}+1\right)\right\}$$

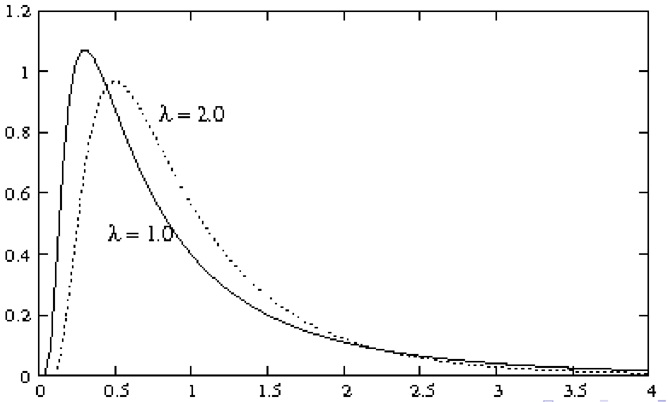
Inverse Gaussian Distribution: PDF

- The shape of the Inverse Gaussian PDF for different λ.



Inverse Gaussian Distribution: PDF

- The shape of the Inverse Gaussian PDF for different λ.



Skew normal distribution: Introduction

So far skewed distribution has been obtained for non-negative random variable. Now we provide a method to introduce skewness to a random variable which may have a support on the entire real line also.

- Consider two indepedent standard normal independent random variables, say X and Y.
- Then because of symmetry

$$P(Y < X) = \frac{1}{2}$$

Skew normal distribution: Introduction

So far skewed distribution has been obtained for non-negative random variable. Now we provide a method to introduce skewness to a random variable which may have a support on the entire real line also.

- Consider two indepedent standard normal independent random variables, say X and Y.

- Then because of symmetry

$$P(Y < X) = \frac{1}{2}$$

Skew normal distribution: Introduction

So far skewed distribution has been obtained for non-negative random variable. Now we provide a method to introduce skewness to a random variable which may have a support on the entire real line also.

- Consider two indepedent standard normal independent random variables, say X and Y.

- Then because of symmetry

$$P(Y < X) = \frac{1}{2}$$

Skew normal distribution: Introduction

- Now suppose $\alpha > 0$ and consider $P(Y < \alpha X)$.
- In this case also since $Y - \alpha X$ is a normal random variable with mean 0, then clearly $P(Y < \alpha X) = 1/2$
- On the otherhand

$$P(Y < \alpha X) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x)dx = \frac{1}{2}.$$

- Therefore,

$$\int_{-\infty}^{\infty} 2\phi(x)\Phi(\alpha x) = 1.$$

Skew normal distribution: Introduction

- Now suppose $\alpha > 0$ and consider $P(Y < \alpha X)$.
- In this case also since $Y - \alpha X$ is a normal random variable with mean 0, then clearly $P(Y < \alpha X) = 1/2$
- On the otherhand

$$P(Y < \alpha X) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x) dx = \frac{1}{2}.$$

- Therefore,

$$\int_{-\infty}^{\infty} 2\phi(x)\Phi(\alpha x) = 1.$$

Skew normal distribution: Introduction

- Now suppose $\alpha > 0$ and consider $P(Y < \alpha X)$.
- In this case also since $Y - \alpha X$ is a normal random variable with mean 0, then clearly $P(Y < \alpha X) = 1/2$
- On the otherhand

$$P(Y < \alpha X) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x)dx = \frac{1}{2}.$$

- Therefore,

$$\int_{-\infty}^{\infty} 2\phi(x)\Phi(\alpha x) = 1.$$

Skew normal distribution: Introduction

- Now suppose $\alpha > 0$ and consider $P(Y < \alpha X)$.
- In this case also since $Y - \alpha X$ is a normal random variable with mean 0, then clearly $P(Y < \alpha X) = 1/2$
- On the otherhand

$$P(Y < \alpha X) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x) dx = \frac{1}{2}.$$

- Therefore,

$$\int_{-\infty}^{\infty} 2\phi(x)\Phi(\alpha x) = 1.$$

Skew normal distribution: Introduction

- Now suppose $\alpha > 0$ and consider $P(Y < \alpha X)$.
- In this case also since $Y - \alpha X$ is a normal random variable with mean 0, then clearly $P(Y < \alpha X) = 1/2$
- On the otherhand

$$P(Y < \alpha X) = \int_{-\infty}^{\infty} \phi(x)\Phi(\alpha x)dx = \frac{1}{2}.$$

- Therefore,

$$\int_{-\infty}^{\infty} 2\phi(x)\Phi(\alpha x) = 1.$$

Skew normal distribution

- Skew normal distribution has the following PDF:

$$f(x;\alpha) = 2\phi(x)\Phi(\alpha x); \qquad -\infty < x < \infty$$

- It is a skewed distribution on the whole real line

Skew normal distribution

- Skew normal distribution has the following PDF:

$$f(x;\alpha) = 2\phi(x)\Phi(\alpha x); \qquad -\infty < x < \infty$$

- It is a skewed distribution on the whole real line

Skew normal distribution

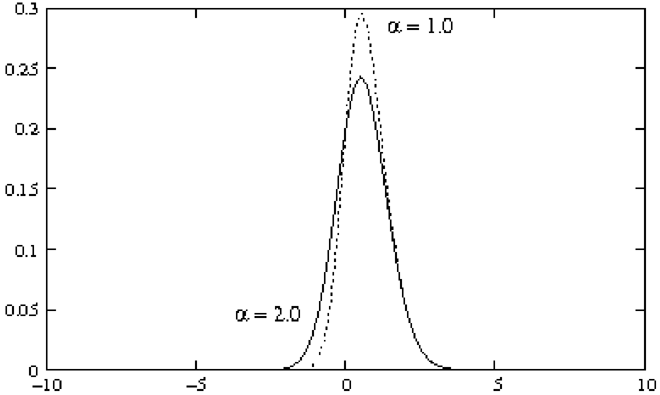
- Skew normal distribution has the following PDF:

$$f(x;\alpha) = 2\phi(x)\Phi(\alpha x); \qquad -\infty < x < \infty$$

- It is a skewed distribution on the whole real line

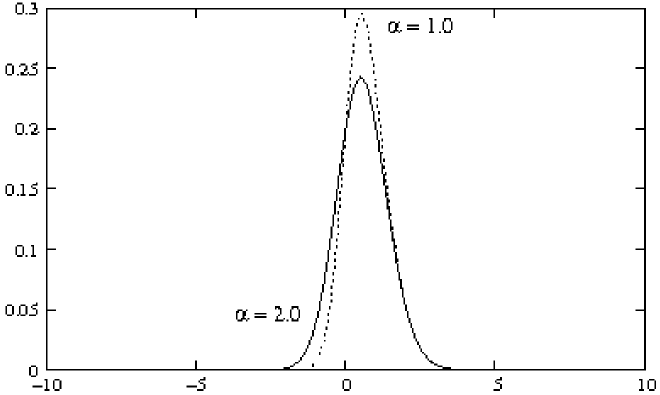
Skew Normal Random Variable

- The shape of the skew normal PDF for different α.



Skew Normal Random Variable

- The shape of the skew normal PDF for different α.



Skewed Distribution

Note that the way skewness has been introduced for skew normal distribution, it can be used for many other cases also.

- Suppose X and Y are two independent idetically distributed random variables, with PDF and CDF as $f(\cdot)$ and $F(\cdot)$ respectively.
- If $c = P(Y < \alpha X)$, then clearly

$$g(x) = \frac{1}{c} f(x) F(\alpha x)$$

is a proper density function.

Skewed Distribution

Note that the way skewness has been introduced for skew normal distribution, it can be used for many other cases also.

- Suppose X and Y are two independent idetically distributed random variables, with PDF and CDF as $f(\cdot)$ and $F(\cdot)$ respectively.
- If $c = P(Y < \alpha X)$, then clearly

$$g(x) = \frac{1}{c} f(x) F(\alpha x)$$

is a proper density function.

Skewed Distribution

Note that the way skewness has been introduced for skew normal distribution, it can be used for many other cases also.

- Suppose X and Y are two independent idetically distributed random variables, with PDF and CDF as $f(\cdot)$ and $F(\cdot)$ respectively.
- If $c = P(Y < \alpha X)$, then clearly

$$g(x) = \frac{1}{c} f(x) F(\alpha x)$$

is a proper density function.

Skewed distribution: Example

If c can be calculated easily then the skewed distribution can be used quite easily in practice.

- For example take $f(x) = e^{-x}$; $x > 0$. In this case

$$c = \int_0^\infty (1 - e^{-\alpha x}) e^{-x} dx = 1 - \frac{1}{\alpha + 1}.$$

- Therefore,

$$g(x) = \frac{\alpha + 1}{\alpha} e^{-x}(1 - e^{-\alpha x}); \quad x > 0$$

is a proper density function.

Skewed distribution: Example

If c can be calculated easily then the skewed distribution can be used quite easily in practice.

- For example take $f(x) = e^{-x}; \; x > 0$. In this case

$$c = \int_0^\infty (1 - e^{-\alpha x}) e^{-x} dx = 1 - \frac{1}{\alpha + 1}.$$

- Therefore,

$$g(x) = \frac{\alpha + 1}{\alpha} e^{-x} (1 - e^{-\alpha x}); \quad x > 0$$

is a proper density function.

Skewed distribution: Example

If c can be calculated easily then the skewed distribution can be used quite easily in practice.

- For example take $f(x) = e^{-x}; \; x > 0$. In this case

$$c = \int_0^\infty (1 - e^{-\alpha x}) e^{-x} dx = 1 - \frac{1}{\alpha + 1}.$$

- Therefore,

$$g(x) = \frac{\alpha + 1}{\alpha} e^{-x}(1 - e^{-\alpha x}); \quad x > 0$$

is a proper density function.

Outline

Birnbaum-Saunders distribution: PDF and CDF

- Birnbaum-Saunders Distribution has the following CDF:

$$F(x;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{\beta}{x}\right)^{1/2}\right\}\right]$$

- It is a skewed distribution on the positive real line.
- The PDF of Birnbaum-Saunders distribution is.

$$f(x) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{1/2} + \left(\frac{\beta}{x}\right)^{3/2}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right]$$

Birnbaum-Saunders distribution: PDF and CDF

- Birnbaum-Saunders Distribution has the following CDF:

$$F(x;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{\beta}{x}\right)^{1/2}\right\}\right]$$

- It is a skewed distribution on the positive real line.
- The PDF of Birnbaum-Saunders distribution is.

$$f(x) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{1/2} + \left(\frac{\beta}{x}\right)^{3/2}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right]$$

Birnbaum-Saunders distribution: PDF and CDF

- Birnbaum-Saunders Distribution has the following CDF:

$$F(x; \alpha, \beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{\beta}{x}\right)^{1/2}\right\}\right]$$

- It is a skewed distribution on the positive real line.
- The PDF of Birnbaum-Saunders distribution is.

$$f(x) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{1/2} + \left(\frac{\beta}{x}\right)^{3/2}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right]$$

Birnbaum-Saunders distribution: PDF and CDF

- Birnbaum-Saunders Distribution has the following CDF:

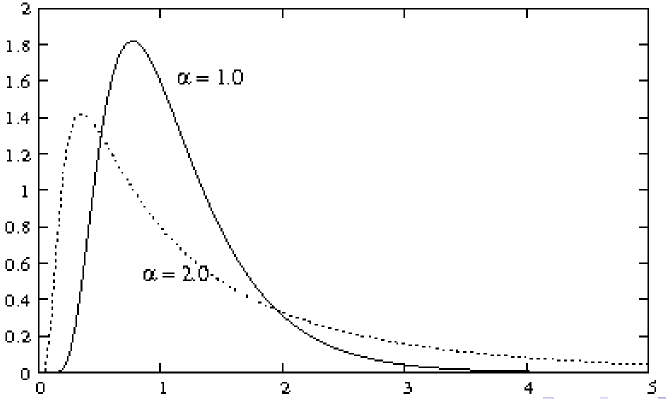
$$F(x;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{\beta}{x}\right)^{1/2}\right\}\right]$$

- It is a skewed distribution on the positive real line.
- The PDF of Birnbaum-Saunders distribution is.

$$f(x) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{1/2} + \left(\frac{\beta}{x}\right)^{3/2}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right]$$

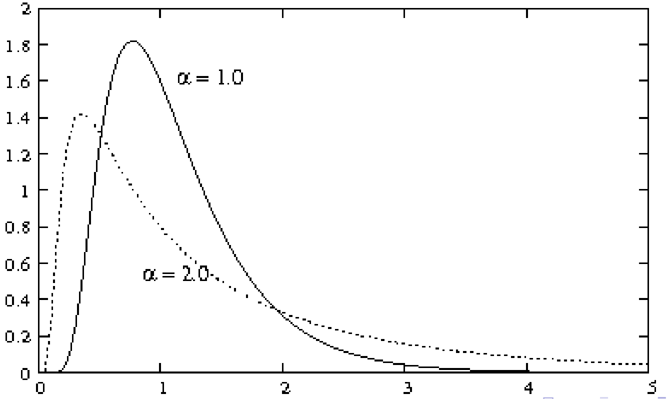
Birnbaum-Saunders Random Variable

- The shape of the Birnbaum-Saunders PDF for different α.



Birnbaum-Saunders Random Variable

- The shape of the Birnbaum-Saunders PDF for different α.



Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- Birnbaum-Saunders distribution has been developed to model failures due to crack.
- It is a assumed that the j-th cycle leads to an increase in crack X_j amount.
- It is further assumed that $\sum_{j=1}^{n} X_j$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.
- Then the probability that the crack does not exceed a critical length ω is
- $$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right)$$

Birnbaum-Saunders distribution

- It is further assumed that the failure occurs when the crack length exceeds ω.
- If T denotes the lifetime (in number of cycles) until failure, then the CDF of T is approximately
- $$P(T \le t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right).$$
- It is exactly the same form as defined before.

Birnbaum-Saunders distribution

- It is further assumed that the failure occurs when the crack length exceeds ω.
- If T denotes the lifetime (in number of cycles) until failure, then the CDF of T is approximately
- $$P(T \le t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right).$$
- It is exactly the same form as defined before.

Birnbaum-Saunders distribution

- It is further assumed that the failure occurs when the crack length exceeds ω.
- If T denotes the lifetime (in number of cycles) until failure, then the CDF of T is approximately
- $$P(T \le t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right).$$
- It is exactly the same form as defined before.

Birnbaum-Saunders distribution

- It is further assumed that the failure occurs when the crack length exceeds ω.
- If T denotes the lifetime (in number of cycles) until failure, then the CDF of T is approximately
- $$P(T \le t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right).$$
- It is exactly the same form as defined before.

Birnbaum-Saunders distribution

- It is further assumed that the failure occurs when the crack length exceeds ω.
- If T denotes the lifetime (in number of cycles) until failure, then the CDF of T is approximately
- $$P(T \leq t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right).$$
- It is exactly the same form as defined before.

Birnbaum-Saunders distribution

- Fatigue failure is due to repeated applications of a common cyclic stress pattern.
- Under the influence of this cyclic stress a dominant crack in the matrial grows until it reaches a critical size w is reached, at that point fatigue failure occurs.
- The crack extention in each cycle are random variables and they are statistically independent.
- The total extension of the crack is approximately normally distributed.

Birnbaum-Saunders distribution

- Fatigue failure is due to repeated applications of a common cyclic stress pattern.
- Under the influence of this cyclic stress a dominant crack in the matrial grows until it reaches a critical size w is reached, at that point fatigue failure occurs.
- The crack extention in each cycle are random variables and they are statistically independent.
- The total extension of the crack is approximately normally distributed.

Birnbaum-Saunders distribution

- Fatigue failure is due to repeated applications of a common cyclic stress pattern.
- Under the influence of this cyclic stress a dominant crack in the matrial grows until it reaches a critical size w is reached, at that point fatigue failure occurs.
- The crack extention in each cycle are random variables and they are statistically independent.
- The total extension of the crack is approximately normally distributed.

Birnbaum-Saunders distribution

- Fatigue failure is due to repeated applications of a common cyclic stress pattern.
- Under the influence of this cyclic stress a dominant crack in the matrial grows until it reaches a critical size w is reached, at that point fatigue failure occurs.
- The crack extention in each cycle are random variables and they are statistically independent.
- The total extension of the crack is approximately normally distributed.

Birnbaum-Saunders distribution

- Fatigue failure is due to repeated applications of a common cyclic stress pattern.
- Under the influence of this cyclic stress a dominant crack in the matrial grows until it reaches a critical size w is reached, at that point fatigue failure occurs.
- The crack extention in each cycle are random variables and they are statistically independent.
- The total extension of the crack is approximately normally distributed.

Birnbaum-Saunders distribution

Desmond (1985, 1986) made the following observations:

- A variety of distributions for crack size are possible which still result in a Birnbaum-Saunders distribution for the fatigue failure time.
- It is possible to allow the crack increment in a given cycle to depend on the total crack size at the beginning of the cycle, and still obtain a fatigue failure life distribution as the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

Desmond (1985, 1986) made the following observations:

- A variety of distributions for crack size are possible which still result in a Birnbaum-Saunders distribution for the fatigue failure time.

- It is possible to allow the crack increment in a given cycle to depend on the total crack size at the beginning of the cycle, and still obtain a fatigue failure life distribution as the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

Desmond (1985, 1986) made the following observations:

- A variety of distributions for crack size are possible which still result in a Birnbaum-Saunders distribution for the fatigue failure time.
- It is possible to allow the crack increment in a given cycle to depend on the total crack size at the beginning of the cycle, and still obtain a fatigue failure life distribution as the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

The following observations are useful:

- If T a Birnbaum-Saunders distribution, say, BS(α, β) then consider the following transfomation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right]$$

- Equivalently

$$T = \beta\left(1 + 2X^2 + 2X\left(1 + X^2\right)^{1/2}\right)$$

- Then X is normally distributed with mean zero and variance $\alpha^2/4$.

Birnbaum-Saunders distribution

The following observations are useful:

- If T a Birnbaum-Saunders distribution, say, BS(α, β) then consider the following transfomation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right]$$

- Equivalently

$$T = \beta\left(1 + 2X^2 + 2X\left(1 + X^2\right)^{1/2}\right)$$

- Then X is normally distributed with mean zero and variance $\alpha^2/4$.

Birnbaum-Saunders distribution

The following observations are useful:

- If T a Birnbaum-Saunders distribution, say, BS(α, β) then consider the following transfomation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right]$$

- Equivalently

$$T = \beta\left(1 + 2X^2 + 2X\left(1 + X^2\right)^{1/2}\right)$$

- Then X is normally distributed with mean zero and variance $\alpha^2/4$.

Birnbaum-Saunders distribution

The following observations are useful:

- If T a Birnbaum-Saunders distribution, say, BS(α, β) then consider the following transfomation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right]$$

- Equivalently

$$T = \beta\left(1 + 2X^2 + 2X\left(1 + X^2\right)^{1/2}\right)$$

- Then X is normally distributed with mean zero and variance $\alpha^2/4$.

Birnbaum-Saunders distribution

The above transformation becomes very helpful:

- It can be used very easily to generate samples from Birnbaum-Saunders distribution.
- It helps to derive different moments of the Birnbaum-Saunders distribution.
- It helps to derive some other properties of the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

The above transformation becomes very helpful:

- It can be used very easily to generate samples from Birnbaum-Saunders distribution.
- It helps to derive different moments of the Birnbaum-Saunders distribution.
- It helps to derive some other properties of the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

The above transformation becomes very helpful:

- It can be used very easily to generate samples from Birnbaum-Saunders distribution.
- It helps to derive different moments of the Birnbaum-Saunders distribution.
- It helps to derive some other properties of the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution

The above transformation becomes very helpful:

- It can be used very easily to generate samples from Birnbaum-Saunders distribution.
- It helps to derive different moments of the Birnbaum-Saunders distribution.
- It helps to derive some other properties of the Birnbaum-Saunders distribution.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Basic Properties

- Here α is the shape parameter and β is the scale parameter.
- α governs the shape of PDF and hazard function.
- For all values of α the PDF is unimodal.
- Mode cannot be obtained in explcit form, it has to be obtained by solving a non-linear equation in α.
- Clearly, the median is at β, for all α.

Birnbaum-Saunders distribution: Moments

- Mean:

$$E(T) = \beta \left(1 + \frac{\alpha^2}{2}\right)$$

- Variance

$$\text{Var}(T) = (\alpha\beta)^2 \left(1 + \frac{5}{4}\alpha^2\right)$$

- Skewness

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}$$

- Kurtosis

$$3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}$$

Birnbaum-Saunders distribution: Moments

- Mean:

$$E(T) = \beta\left(1 + \frac{\alpha^2}{2}\right)$$

- Variance

$$\text{Var}(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right)$$

- Skewness

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}$$

- Kurtosis

$$3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}$$

Birnbaum-Saunders distribution: Moments

- Mean:

$$E(T) = \beta\left(1 + \frac{\alpha^2}{2}\right)$$

- Variance

$$\text{Var}(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right)$$

- Skewness

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2+6)}{(5\alpha^2+4)^3}$$

- Kurtosis

$$3 + \frac{6\alpha^2(93\alpha^2+41)}{(5\alpha^2+4)^2}$$

Birnbaum-Saunders distribution: Moments

- Mean:

$$E(T) = \beta\left(1 + \frac{\alpha^2}{2}\right)$$

- Variance

$$\text{Var}(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right)$$

- Skewness

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}$$

- Kurtosis

$$3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}$$

Birnbaum-Saunders distribution: Moments

- Mean:

$$E(T) = \beta\left(1 + \frac{\alpha^2}{2}\right)$$

- Variance

$$\text{Var}(T) = (\alpha\beta)^2\left(1 + \frac{5}{4}\alpha^2\right)$$

- Skewness

$$\beta_1(T) = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}$$

- Kurtosis

$$3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}$$

Birnbaum-Saunders distribution: Inverse

Anothe interesting observation which is useful for estimation purposes:

- If T is a Birnbaum-Saunders distribution, *i.e.* BS(α, β), then T^{-1} is also a Birnbaum-Saunders with parameters α and β^{-1}
- The above observation is very useful. Immediately we obtain
-

$$E(T^{-1}) = \beta^{-1}(1 + \frac{1}{2}\alpha^2), \quad \text{Var}(T) = \alpha^2\beta^{-2}(1 + \frac{5}{4}\alpha^2).$$

Birnbaum-Saunders distribution: Inverse

Anothe interesting observation which is useful for estimation purposes:

- If T is a Birnbaum-Saunders distribution, *i.e.* BS(α, β), then T^{-1} is also a Birnbaum-Saunders with parameters α and β^{-1}
- The above observation is very useful. Immediately we obtain
-

$$E(T^{-1}) = \beta^{-1}(1 + \frac{1}{2}\alpha^2), \quad \text{Var}(T) = \alpha^2\beta^{-2}(1 + \frac{5}{4}\alpha^2).$$

Birnbaum-Saunders distribution: Inverse

Anothe interesting observation which is useful for estimation purposes:

- If T is a Birnbaum-Saunders distribution, *i.e.* BS(α, β), then T^{-1} is also a Birnbaum-Saunders with parameters α and β^{-1}
- The above observation is very useful. Immediately we obtain
- $$E(T^{-1}) = \beta^{-1}(1 + \frac{1}{2}\alpha^2), \qquad \text{Var}(T) = \alpha^2\beta^{-2}(1 + \frac{5}{4}\alpha^2).$$

Birnbaum-Saunders distribution: Inverse

Anothe interesting observation which is useful for estimation purposes:

- If T is a Birnbaum-Saunders distribution, *i.e.* BS(α, β), then T^{-1} is also a Birnbaum-Saunders with parameters α and β^{-1}
- The above observation is very useful. Immediately we obtain
- $$E(T^{-1}) = \beta^{-1}(1 + \frac{1}{2}\alpha^2), \qquad \text{Var}(T) = \alpha^2\beta^{-2}(1 + \frac{5}{4}\alpha^2).$$

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- If X is a random vaiable with the PDF $f(x)$, CDF $F(x)$, and support on $(0,\infty)$, then the hazard function of X is defined as

$$h(x) = \frac{f(x)}{1-F(x)} = \lim_{h \downarrow 0} P(x < X < x+h | X \geq x)$$

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- If X is a random vaiable with the PDF $f(x)$, CDF $F(x)$, and support on $(0, \infty)$, then the hazard function of X is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} = \lim_{h \downarrow 0} P(x < X < x + h | X \geq x)$$

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- If X is a random vaiable with the PDF $f(x)$, CDF $F(x)$, and support on $(0, \infty)$, then the hazard function of X is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} = \lim_{h \downarrow 0} P(x < X < x + h | X \geq x)$$

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- If X is a random vaiable with the PDF $f(x)$, CDF $F(x)$, and support on $(0, \infty)$, then the hazard function of X is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} = \lim_{h \downarrow 0} P(x < X < x + h | X \geq x)$$

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.
- Hazard function is difficult to estimate, but there are some empirical data analysis methods which can be used to determine the shape of the hazard function.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.
- Hazard function is difficult to estimate, but there are some empirical data analysis methods which can be used to determine the shape of the hazard function.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.
- Hazard function is difficult to estimate, but there are some empirical data analysis methods which can be used to determine the shape of the hazard function.

Birnbaum-Saunders distribution: Hazard Function

Hazard function plays an important role in lifetime data analysis.

- It uniquely defines the distribution function.
- Often the prior information of the hazard is used to model the data.
- Hazard function is difficult to estimate, but there are some empirical data analysis methods which can be used to determine the shape of the hazard function.

Birnbaum-Saunders distribution: Hazard Function

- The shape of the hazard function of Birnbaum-Saunders distribution is unimodal.
- The turning point of the hazard function can be obtained by solving a non-linear equation involving α.
- The tunring point of the hazard function of the Birnbaum-Saumnders distribution can be approximated very well for $\alpha > 0.25$, by

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

- The approximation works very well for $\alpha > 0.6$.

Birnbaum-Saunders distribution: Hazard Function

- The shape of the hazard function of Birnbaum-Saunders distribution is unimodal.
- The turning point of the hazard function can be obtained by solving a non-linear equation involving α.
- The tunring point of the hazard function of the Birnbaum-Saumnders distribution can be approximated very well for $\alpha > 0.25$, by

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

- The approximation works very well for $\alpha > 0.6$.

Birnbaum-Saunders distribution: Hazard Function

- The shape of the hazard function of Birnbaum-Saunders distribution is unimodal.
- The turning point of the hazard function can be obtained by solving a non-linear equation involving α.
- The tunring point of the hazard function of the Birnbaum-Saumnders distribution can be approximated very well for $\alpha > 0.25$, by

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

- The approximation works very well for $\alpha > 0.6$.

Birnbaum-Saunders distribution: Hazard Function

- The shape of the hazard function of Birnbaum-Saunders distribution is unimodal.
- The turning point of the hazard function can be obtained by solving a non-linear equation involving α.
- The tunring point of the hazard function of the Birnbaum-Saumnders distribution can be approximated very well for $\alpha > 0.25$, by

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

- The approximation works very well for $\alpha > 0.6$.

Birnbaum-Saunders distribution: Hazard Function

- The shape of the hazard function of Birnbaum-Saunders distribution is unimodal.
- The turning point of the hazard function can be obtained by solving a non-linear equation involving α.
- The tunring point of the hazard function of the Birnbaum-Saumnders distribution can be approximated very well for $\alpha > 0.25$, by

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

- The approximation works very well for $\alpha > 0.6$.

Outline

Birnbaum-Saunders distribution: Estimation

Let $\{t_1, \cdots, t_n\}$ be a sample of size n from BS(α, β). Now we will discuss different estimation procedures of α and β. First we will discuss the maximum likelihood etimators.

- The sample arithmatic and harmonic means are quite important in this case, and we will define them as;

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

- Then it can be seen that for a given β, $\widehat{\alpha}(\beta)$ maximizes the likelihood function, when

$$\widehat{\alpha}(\beta) = \left[\frac{s}{\widehat{\beta}} + \frac{\widehat{\beta}}{r} - 2\right]^{1/2}$$

Birnbaum-Saunders distribution: Estimation

Let $\{t_1, \cdots, t_n\}$ be a sample of size n from BS(α, β). Now we will discuss different estimation procedures of α and β. First we will discuss the maximum likelihood etimators.

- The sample arithmatic and harmonic means are quite important in this case, and we will define them as;

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

- Then it can be seen that for a given β, $\widehat{\alpha}(\beta)$ maximizes the likelihood function, when

$$\widehat{\alpha}(\beta) = \left[\frac{s}{\widehat{\beta}} + \frac{\widehat{\beta}}{r} - 2\right]^{1/2}$$

Birnbaum-Saunders distribution: Estimation

Let $\{t_1, \cdots, t_n\}$ be a sample of size n from BS(α, β). Now we will discuss different estimation procedures of α and β. First we will discuss the maximum likelihood etimators.

- The sample arithmatic and harmonic means are quite important in this case, and we will define them as;

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}.$$

- Then it can be seen that for a given β, $\widehat{\alpha}(\beta)$ maximizes the likelihood function, when

$$\widehat{\alpha}(\beta) = \left[\frac{s}{\widehat{\beta}} + \frac{\widehat{\beta}}{r} - 2\right]^{1/2}$$

Birnbaum-Saunders distribution: Estimation

- Finally the MLE of β can be obtained by finding the postive root of the following non-linear equation

$$\beta^2 - \beta(2r + K(\beta)) + r(s + K(\beta))c(\alpha) = 0,$$

- Here $K(x)$ is the harmonic mean function defined by

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1}; \quad x \geq 0.$$

Birnbaum-Saunders distribution: Estimation

- Finally the MLE of β can be obtained by finding the postive root of the following non-linear equation

$$\beta^2 - \beta(2r + K(\beta)) + r(s + K(\beta))c(\alpha) = 0,$$

- Here $K(x)$ is the harmonic mean function defined by

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1}; \quad x \geq 0.$$

Birnbaum-Saunders distribution: Estimation

- Finally the MLE of β can be obtained by finding the postive root of the following non-linear equation

$$\beta^2 - \beta(2r + K(\beta)) + r(s + K(\beta))c(\alpha) = 0,$$

- Here $K(x)$ is the harmonic mean function defined by

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x + t_i)^{-1}\right]^{-1}; \quad x \geq 0.$$

Birnbaum-Saunders distribution: Estimation

The asymptotic joint distribution of α and β is bivariate normal and is given by

$$\sqrt{n}\left(\widehat{\alpha}-\alpha, \widehat{\beta}-\beta\right) \xrightarrow{d} \begin{bmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & \frac{\beta^2}{0.25+\alpha^{-2}+I(\alpha)} \end{bmatrix}$$

where

$$I(\alpha) = 2\int_0^\infty ((1+g(\alpha x))^{-1} - 0.5)^2 d\Phi(x)$$

$$g(y) = 1 + \frac{y^2}{2} + y\left(1+\frac{y^2}{4}\right)^{1/2}$$

Birnbaum-Saunders distribution: Modified Moment Estimation

To avoid solving the non-linear equation, moment type estimators of α and β have been proposed, and they can be obtained in explicit forms. It is basically obtained by equating $E(T)$ and $E(T^{-1})$ with the arithmatic mean and the harmonic mean of the data. They are as folllows;

$$\widetilde{\alpha} = \left\{ 2\left[\left(\frac{s}{r} \right)^{1/2} - 1 \right] \right\}^{1/2}, \quad \widetilde{\beta} = (sr)^{1/2}$$

Birnbaum-Saunders distribution: Asymptotic Distribution

The asymptotic joint distribution of $\widetilde{\alpha}$ and $\widetilde{\beta}$ is also bivariate normal and is given by

$$\sqrt{n}\left(\widetilde{\alpha}-\alpha,\widetilde{\beta}-\beta\right) \xrightarrow{d} \begin{bmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & (\alpha\beta)^2 \frac{1+0.75\alpha^2}{(1+0.5\alpha^{-2})^2} \end{bmatrix}$$

Birnbaum-Saunders distribution: Comparison

The following observations can be easily made:
The asymptotic distributions of the MLE of α and MME of α are same.

The asymptotic distributions of the MLE of λ and MME of α are different. It might be nice to compare the asymptotic variances of the estimators.

Another interesting problem would be to compare the asymptotic variances of the percentile estimators based on MLE or MME.

Birnbaum-Saunders distribution: Open Problems

- Although extensive work has been done on two-parameter Birnbaum-Saunders distribution, but not much work has been done on the three-parameter Birnbaum-Saunders ditribution. Suppose T is a Birnbaum-Saunders distribution consider the random variable $Y = T - \mu$, *i.e.* μ is the location parameter. Then Y has the three-parameter Birnbaum-Saunders distribution. It will be intersting to find the MLEs and study their properties.
- It will be interesting to study the stress-strength parameter, $R = P(X < Y)$
- Another interesting problem to discriminate the Birnbaum-Saunders distribution and some other related

Birnbaum-Saunders distribution: Open Problems

- Although extensive work has been done on two-parameter Birnbaum-Saunders distribution, but not much work has been done on the three-parameter Birnbaum-Saunders ditribution. Suppose T is a Birnbaum-Saunders distribution consider the random variable $Y = T - \mu$, *i.e.* μ is the location parameter. Then Y has the three-parameter Birnbaum-Saunders distribution. It will be intersting to find the MLEs and study their properties.

- It will be interesting to study the stress-strength parameter, $R = P(X < Y)$

- Another interesting problem to discriminate the Birnbaum-Saunders distribution and some other related

Birnbaum-Saunders distribution: Open Problems

- Although extensive work has been done on two-parameter Birnbaum-Saunders distribution, but not much work has been done on the three-parameter Birnbaum-Saunders ditribution. Suppose T is a Birnbaum-Saunders distribution consider the random variable $Y = T - \mu$, *i.e.* μ is the location parameter. Then Y has the three-parameter Birnbaum-Saunders distribution. It will be intersting to find the MLEs and study their properties.

- It will be interesting to study the stress-strength parameter, $R = P(X < Y)$

- Another interesting problem to discriminate the Birnbaum-Saunders distribution and some other related

Birnbaum-Saunders distribution: Open Problems

- Although extensive work has been done on two-parameter Birnbaum-Saunders distribution, but not much work has been done on the three-parameter Birnbaum-Saunders ditribution. Suppose T is a Birnbaum-Saunders distribution consider the random variable $Y = T - \mu$, $i.e.$ μ is the location parameter. Then Y has the three-parameter Birnbaum-Saunders distribution. It will be intersting to find the MLEs and study their properties.

- It will be interesting to study the stress-strength parameter, $R = P(X < Y)$

- Another interesting problem to discriminate the Birnbaum-Saunders distribution and some other related skewed distributions, for example Weibull, log-normal

Outline

Related Distributions: Length Biased BS Distribution

The length biased (LB) version of a particular distribution has received considerable attention in the lifetime data analysis:

- The LB distributions are particular cases of the weighted distributions. If Y is a random variable with PDF $f_Y(\cdot)$, then the weighted version of Y with weight function $w(y)$ has the PDF

$$f_X(x) = \frac{w(x)f_Y(x)}{E(w(Y))},$$

under the assumtion $E(w(Y)) < \infty$.

- If $w(y) = y$, then it is called the LB distribution (LB version of Y).

$$f_T(t) = \frac{tf_Y(t)}{\mu}.$$

Related Distributions: Length Biased BS Distribution

The length biased (LB) version of a particular distribution has received considerable attention in the lifetime data analysis:

- The LB distributions are particular cases of the weighted distributions. If Y is a random variable with PDF $f_Y(\cdot)$, then the weighted version of Y with weight function $w(y)$ has the PDF

$$f_X(x) = \frac{w(x)f_Y(x)}{E(w(Y))},$$

under the assumtion $E(w(Y)) < \infty$.

- If $w(y) = y$, then it is called the LB distribution (LB version of Y).

$$f_T(t) = \frac{tf_Y(t)}{\mu}.$$

Related Distributions: Length Biased BS Distribution

The length biased (LB) version of a particular distribution has received considerable attention in the lifetime data analysis:

- The LB distributions are particular cases of the weighted distributions. If Y is a random variable with PDF $f_Y(\cdot)$, then the weighted version of Y with weight function $w(y)$ has the PDF
$$f_X(x) = \frac{w(x) f_Y(x)}{E(w(Y))},$$
under the assumtion $E(w(Y)) < \infty$.
- If $w(y) = y$, then it is called the LB distribution (LB version of Y).
$$f_T(t) = \frac{t f_Y(t)}{\mu}.$$

Related Distributions: Length Biased BS Distribution

Different properties of the LB version of the Birnbaum-Saunders distribtuion (LBS) can be easily established.

The PDF of LBS can be easily obtained using the standard transformation method.

The CDF can be obtained in terms of $\Phi(\cdot)$.

The mode can be obtained by solving a non-linear equation in α only.

Here also α and β are the shape and scale parameters respectively.

Length Biased BS Distribution: Open Problems

It is observed that the shape of the hazard function of the BS distribution is unimodal, but due to complicated nature of the hazard function it has not yet been established theoretically. It will be nice to have a theoretical proof.

The change point (hazard function) estimation is an important problem. It is important to find an efficient estimation procedure of the change point.

Some approximation along the line of Birnbaum-Saunders distribution will be worth exploring.

Nice representation

Now we will present one nice representation of the Birnbaum-Saunders distribution. We have already defined the Inverse Gaussian random variable which has the following PDF:

$$f(x;\mu,\sigma) = \left(\frac{1}{2\sigma^2\pi x^3}\right)^{1/2} exp\left(-\frac{(x-\mu)^2}{2\sigma^2\mu^2 x}\right)$$

We will denote this as $\text{IG}(\mu,\sigma^2)$.

Suppose $X_1 \sim \text{IG}(\mu,\sigma^2)$, and $X_2^{-1} \sim \text{IG}(\mu^{-1},\sigma^2 mu^2)$, then consider the new random variable for $0 \le p \le 1$:

$$X = \begin{cases} X_1 & \text{w.p} \quad 1-p \\ X_1 & \text{w.p.} \quad p \end{cases}$$

Nice representation

Then the PDF of X can be expressed as follows:

$$f_X(x) = pf_{X_1}(x) + (1-p)f_{X_2}(x)$$

Note that the PDF of X_1 is a Birnbaum-Saunders PDF, and the PDF of X_2 can be easily obtained.

Interestingly, when $p = 1/2$, the PDF of X becomes the PDF of a Birnbaum-Saunders PDF.

Mixture of Birnbaum-Saunders Distributions

The mixture of two Birnbaum-Saunders distributions can be described as follows:

Suppose T_1 and T_2 are two Birnbaum-Saunders distribution, such that $T_1 \sim \text{BS}(\alpha_1, \beta_1)$ and $T_2 \sim \text{BS}(\alpha_2, \beta_2)$. Then consider the random variable Y with the PDF

$$f_Y(t) = p f_{T_1}(t; \alpha_1, \beta_1) + (1-p) f_{T_2}(t; \alpha_2, \beta_2).$$

Mixture of Birnbaum-Saunders Distributions: Properties

Some interesting properties:

The PDF of the mixture can be unimodal or bimodal.

The CDF can be expressed in terms of $\Phi(\cdot)$.

The moments can be obtained in terms of the moments of the Birnbaum-Saunders distributions.

The Hazard function can be written as a mixture of two Birnbaum-Saunders hazard functions.

Mixture of Birnbaum-Saunders Distributions: Estimation

The estimation of four parameters involves the maximization of the log-likelihood function with respecto to four parameters. It becomes a four dimensional optimization problem. To avoid that we can use EM algorithm in this case as follows: Use the following representation:

$$f_Y(t) = \frac{1}{2}\sum_{j=1}^{2} p_j f_{X_1}(t;\mu_j,\sigma_j^2) + \frac{1}{2}\sum_{j=1}^{2} p_j f_{X_2}(t;\mu_j,\sigma_j^2)$$

Outline

Bivariate Birnbaum-Saunders distribution

Remember the univariate Birnbaum-Saunders distribution has been defined as follows:

$$P(T_1 \leq t_1) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right]$$

The bivariate bivariate Birnbaum-Saunders distribution can be defined analogously as follows:

Bivariate Birnbaum-Saunders distribution

Let the joint distribution function of (T_1, T_2) be defined as follows $F(t_1, t_2) = P(T_1 \le t_1, T_2 \le t_2)$, then (T_1, T_2) is said to have bivariate Birnbaum-Saunders distribution with parameters $\alpha_1, \alpha_2, \beta_2, \beta_2, \rho$, if

$$F(t_1, t_2) = \Phi_2 \left[\frac{1}{\alpha_1} \left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}} \right), \frac{1}{\alpha_2} \left(\sqrt{\frac{t_2}{\beta_2}} - \sqrt{\frac{\beta_2}{t_2}} \right); \rho \right]$$

Here $\Phi_2(u, v)$ is the CDF of a standard bivariate normal vector (Z_1, Z_2) with the correlation coefficient ρ. Let's denote this by BVBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$.

Bivariate Birnbaum-Saunders Distribution

Bivariate Birnbaum-Saunders distribution has several interesting properties.

The joint PDF of (T_1, T_2) can be easily obtained interms of the PDF of bivariate normal distribution.

The joint PDF can take different shapes, but it is unimodal, skewed. Need not be symmetric.

The correlation coefficient between T_1 and T_2 can be both positive and negative.

BVBS: Properties

If $(T_1, T_2) \sim \text{BVBS}(\alpha_1, \beta_1, \alpha_2, \beta_2)$, then we have the following results:

$T_1 \sim \text{BS}(\alpha_1, \beta_1)$ and $T_2 \sim \text{BS}(\alpha_2, \beta_2)$.

$(T_1^{-1}, T_2^{-1}) \sim \text{BVBS}(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2^{-1}, \rho)$.

$(T_1^{-1}, T_2) \sim \text{BVBS}(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2, -\rho)$.

$(T_1, T_2^{-1}) \sim \text{BVBS}(\alpha_1, \beta_1, \alpha_2, \beta_2^{-1}, -\rho)$.

Bivariate Birnbaum-Saunders Distribution: Generation

It is very easy to generate Bivariate Birnbaum-Saunders random variables using normal random numbers generator:
Generate first U_1 and U_2 from N(0,1)
Generate

$$Z_1 = \frac{\sqrt{1+\rho}+\sqrt{1-\rho}}{2}U_1 + \frac{\sqrt{1+\rho}-\sqrt{1-\rho}}{2}U_2$$

$$Z_2 = \frac{\sqrt{1+\rho}-\sqrt{1-\rho}}{2}U_1 + \frac{\sqrt{1+\rho}+\sqrt{1-\rho}}{2}U_2$$

Make the transformation:

$$T_i = \beta_i \left[\frac{1}{2}\alpha_i Z_i + \sqrt{\left(\frac{1}{2}\alpha_i Z_i\right)^2 + 1}\right]^2, \quad i = 1, 2.$$

Bivariate Birnbaum-Saunders Distribution: Inference

The MLEs of the unknown parameters can be obtained to solve a five dimensional optimization problem.
The following observation is useful:

$$\left[\left(\sqrt{\frac{T_1}{\beta_1}} - \sqrt{\frac{\beta_1}{T_1}}\right), \left(\sqrt{\frac{T_2}{\beta_2}} - \sqrt{\frac{\beta_2}{T_2}}\right)\right] \sim N_2\left\{(0,0), \Sigma\right\}$$

where

$$\Sigma = \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2\rho \\ \alpha_1\alpha_2\rho & \alpha_2^2 \end{pmatrix}.$$

The MLEs can be obatined by maximixing the profile log-likelihood function.

Thank You