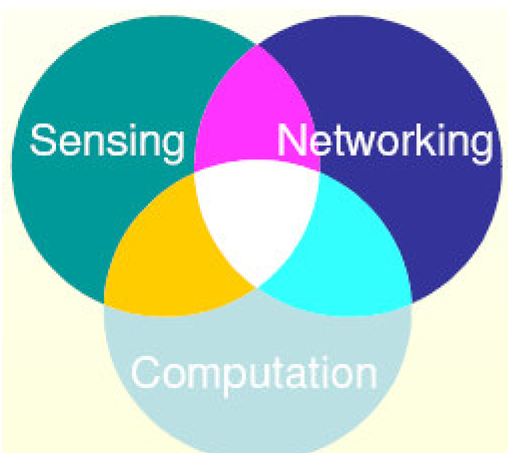
Percolation Theory

Jie Gao
Computer Science Department
Stony Brook University



STONY BROOK
STATE UNIVERSITY OF NEW YORK

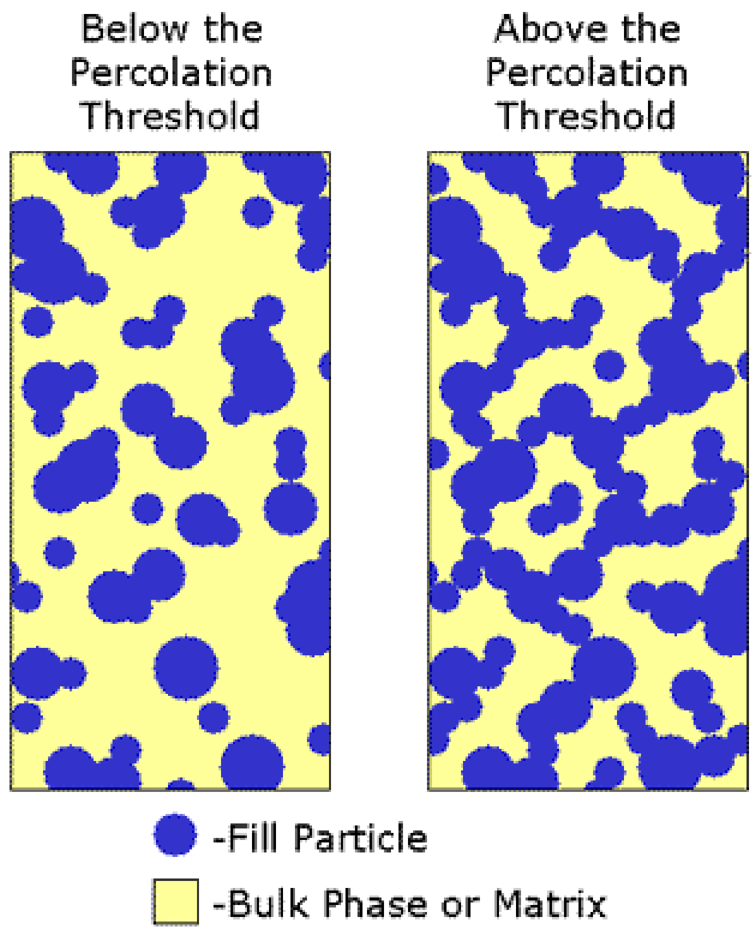
Paper

- Geoffrey Grimmett, **Percolation**, first chapter, Second edition, Springer, 1999.
- E. N. Gilbert, **Random plane networks**. Journal of SIAM 9, 533-543, 1961.
- Massimo Franceschetti, Lorna Booth, Matthew Cook, Ronald Meester, and Jehoshua Bruck, **Continuum percolation with unreliable and spread out connections**, Journal of Statistical Physics, v. 118, N. 3-4, February 2005, pp. 721-734.

On a rainy day

- Observe the raindrops falling on the pavement. Initially the wet regions are isolated and we can find a dry path. Then after some point, the wet regions are connected and we can find a wet path.

- There is a critical density where sudden change happens.



Phase transition

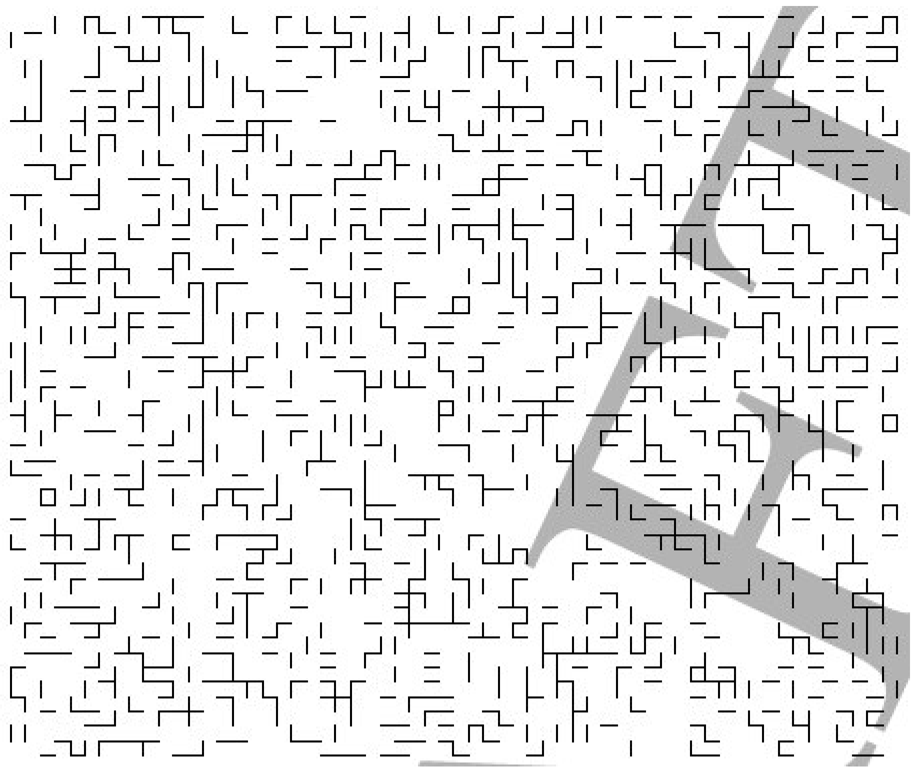
- In physics, a **phase transition** is the transformation of a thermodynamic system from one **phase** to another. The distinguishing characteristic of a **phase transition** is an abrupt sudden change in one or more physical properties, in particular the heat capacity, with a small change in a thermodynamic variable such as the temperature.

- Solid, liquid, and gaseous phases.
- Different magnetic properties.
- Superconductivity of medals.

- This generally stems from the interactions of an extremely large number of particles in a system, and does not appear in systems that are too small.

Bond Percolation

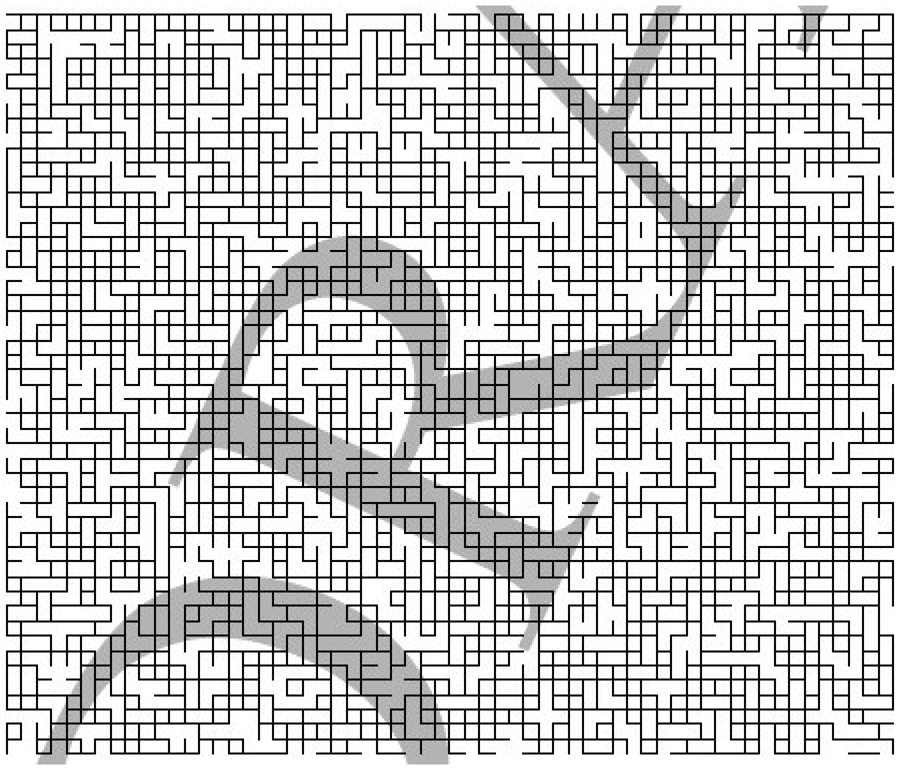
- An infinite grid Z^2, with each link to be “open” (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.25

Bond Percolation

- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.

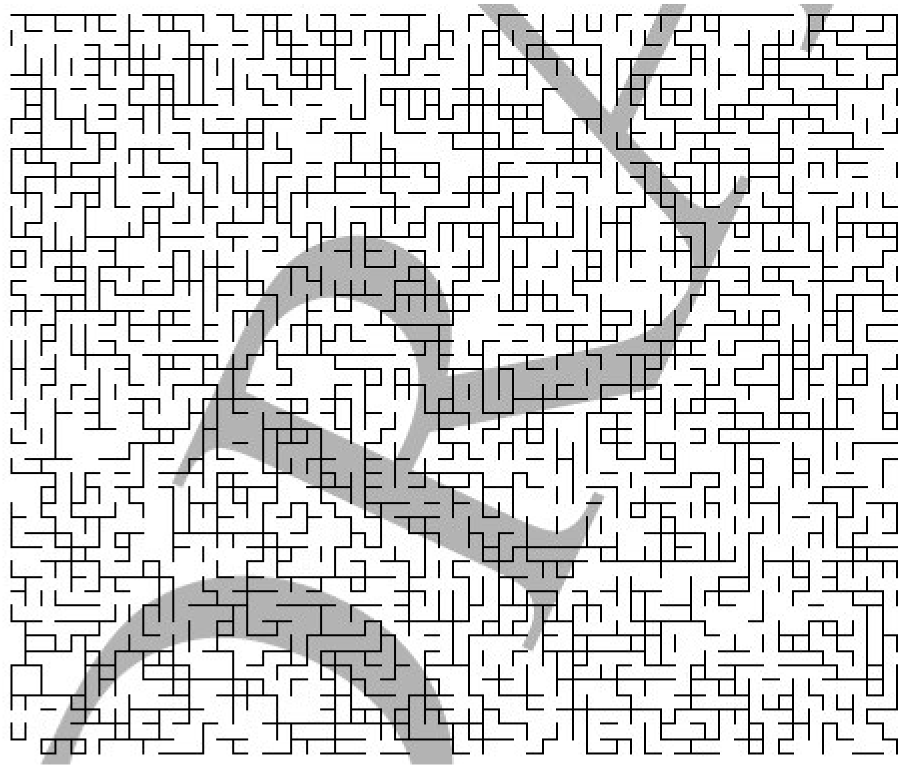


p=0.75

Bond Percolation

- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.

No path from left to right

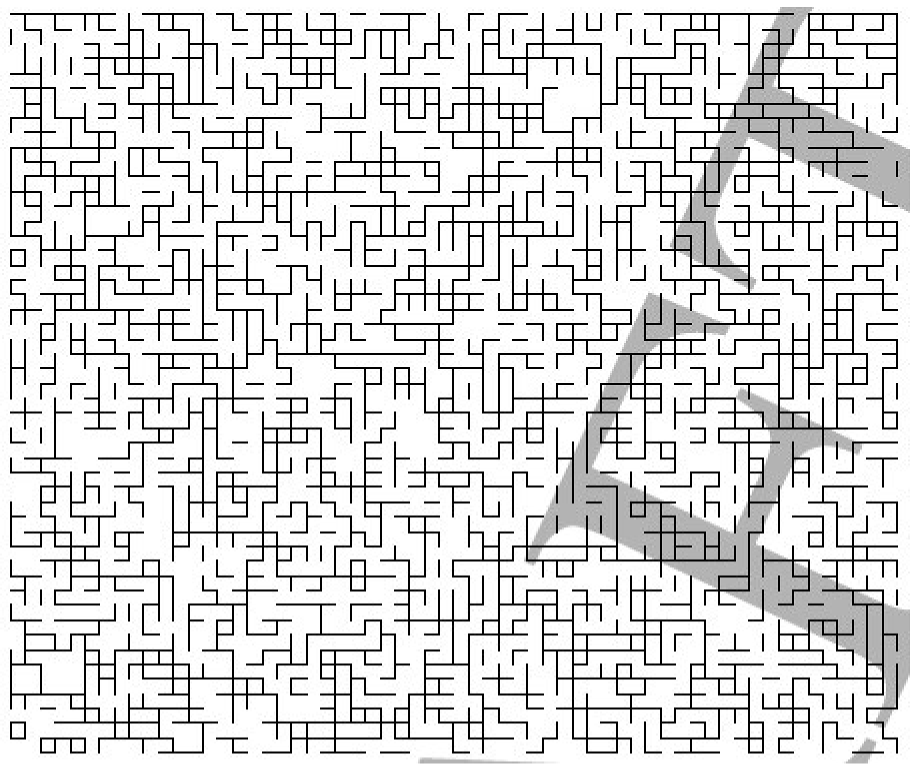


p=0.49

Bond Percolation

- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.

There is a path from left to right!

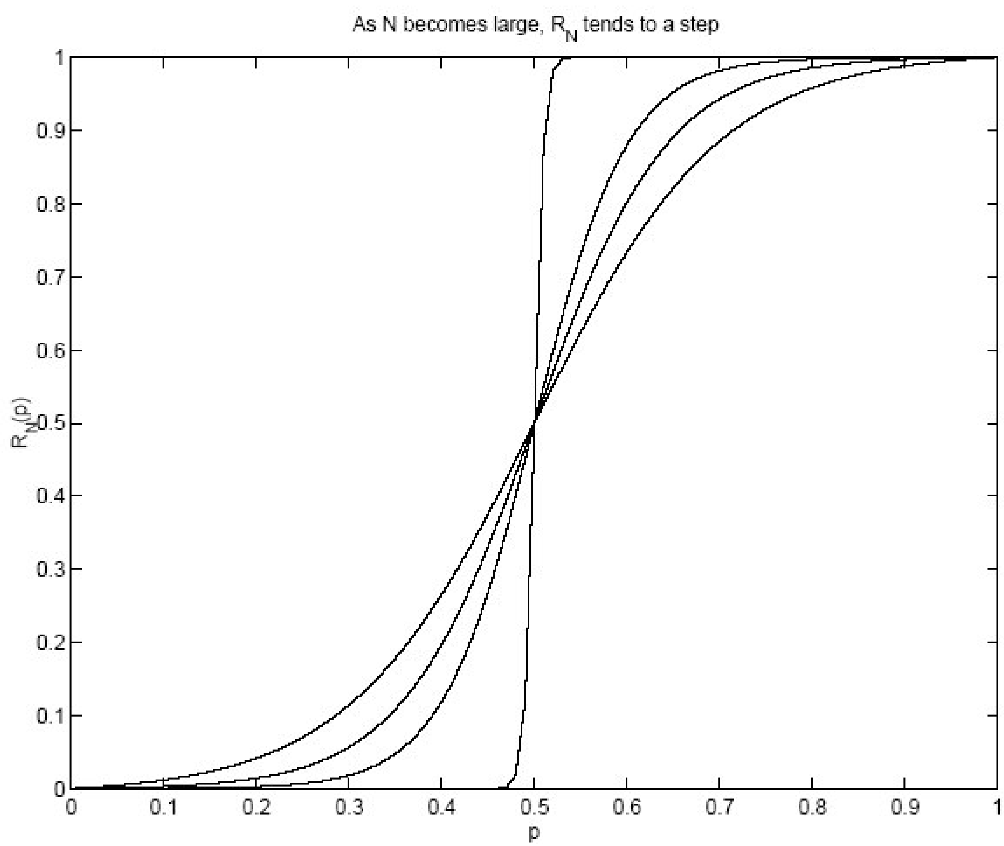


p=0.51

Bond Percolation

- There is a critical threshold p=0.5.

The probability that there is a "bridge" cluster that spans from left to right.



Bond Percolation

- There is a critical threshold $p=0.5$.
- When $p>0.5$, there is a unique infinite size cluster almost always.
- When $p<0.5$, there is no infinitely size cluster.
- When $p=0.5$, the critical value, there is no infinite cluster.

- Percolation theory studies the phase transition in random structures.

Main problems in percolation

- What is the critical threshold for the appearance of some property, e.g., an infinite cluster?
- What is the behavior below the threshold? We know all clusters are finite. How large are they? Distribution of the cluster size?
- What is the behavior above the threshold? We know there exists an infinite cluster? Is it unique? What is the asymptotic size with respect to p and n (the network size)?
- What is the behavior at the threshold? Is there an infinite cluster or not? What is the size of the clusters?

Examples of Percolation

- **Spread of epidemics, virus infection on the Internet.**
  - Each “sick” node has probability p to infect a neighbor node.
  - Denote by p the contagious parameter. If p is above the percolation threshold, then the disease will spread world wide.
  - The real model is more complicated, taking into account the time variation, healing rate, etc.
- **Gossip-based routing, content distribution in P2P network, software upgrade.**
  - The graph is important in deciding the critical value.
  - An interesting result is about the “scale-free” graphs (also called power-law) that model the topology of the Internet or social network: in one of such models (random attachment with preferential rule), the percolation threshold vanishes.

More examples

- **Connectivity of unreliable networks.**
  - Each edge goes down randomly.
  - Is there a path between any two nodes, with high probability?
  - Resilience or fault tolerance of a network to random failures.
- **Random geometric graph, density of wireless nodes (or, critical communication range).**
  - Wireless nodes with Poisson distribution in the plane.
  - Nodes within distance r are connected by an edge.
  - There is a critical threshold on the density (or the communication range) such that the graph has an infinitely large connected component.

Bond percolation

- A grid Z^d, each edge appears with probability p.

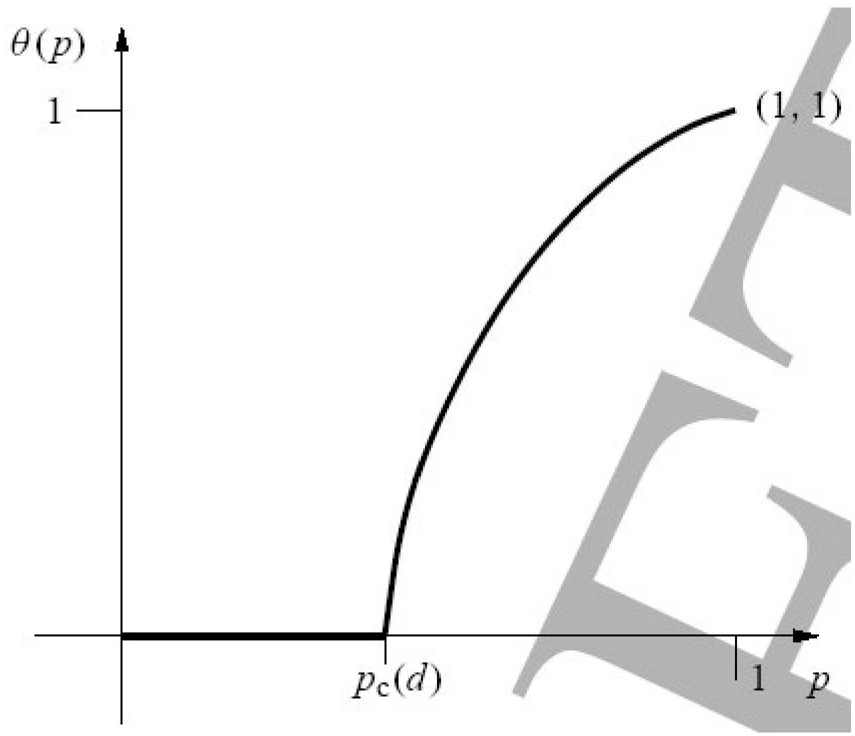
- C(x): the cluster containing the grid node x.
- By symmetry, the shape of C(x) has the same distribution as the shape of C(0), where 0 is the origin.

- $\theta(p)$: the probability that C(0) has infinite size.
- Clearly, when p=0, $\theta(p)=0$, when p=1, $\theta(p)=1$.

- Percolation theory: there exists a threshold $p_c(d)$ such that
  - $\theta(p)>0$, if $p> p_c(d)$;
  - $\theta(p)=0$, if $p< p_c(d)$.

Bond percolation

- This is people's belief on the percolation probability $\theta(p)$, It is known that $\theta(p)$ is a continuous function of p except possibly at the critical probability. However, the possibility of a jump at the critical probability has not been ruled out when $3 \leq d < 19$.



An easy case:1D

- 1D case: a line. Each edge has probability p to be turned on.
- If $p<1$, there are infinitely many missing edges to the left and to the right of the origin. Thus $\theta(p)=0$.
- The threshold $p_c(1) = 1$.
- For general d-dimensional grid Z^d, it can be embedded in the (d+1)-dimensional grid Z^{d+1}.
- Thus if the origin belongs to an infinite cluster in Z^d, it also belongs to an infinite cluster in Z^{d+1}.
- This means: $p_c(d+1) \leq p_c(d)$. In fact it can be proved that $p_c(d+1) < p_c(d)$.

2d: interesting things start to happen

- Theorem: For $d \geq 2$, $0 < p_c(d) < 1$.

- There are 2 phases:
- Subcritical phase, $p < p_c(d)$, $\theta(p)=0$, every vertex is almost surely in a finite cluster. Thus all the clusters are finite.
- Supercritical phase, $p > p_c(d)$, $\theta(p)>0$, every vertex has a strictly positive probability of being in an infinite cluster. Thus there is almost surely at least one infinite cluster.

- At the critical point: this is the most interesting part. Lots of unknowns.
- For $d=2$ or $d \geq 19$, there is no infinite cluster. The problem for the other dimensions is still open.

Critical threshold $p_c(d)$

- We've seen that $p_c(1) = 1$, $p_c(2) = ½$.
- The proof for $p_c(2)$ is non-trivial.
- In fact, the critical values for many percolation processes, even for many regular networks are only approximated by computer simulation.
- We will prove an upper and lower bound for $p_c(2)$.

$$\frac{1}{\lambda(2)} \leq p_{\mathrm{c}}(2) \leq 1 - \frac{1}{\lambda(2)},$$

- $\lambda(d)$: the connective constant.
- $\sigma(n)$: the number of paths starting from origin with length n.

$$\lambda(d) = \lim_{n \to \infty} \{\sigma(n)^{1/n}\}$$

Critical threshold $p_c(d)$

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- $\lambda(d)$: the connective constant.
- $\sigma(n)$: the number of paths starting from origin with length n.

$$\lambda(d) = \lim_{n\to\infty} \{\sigma(n)^{1/n}\}$$

- The exact value of $\lambda(d)$ is unknown for $d \ge 2$. But there is an easy upper bound $\lambda(d) \le 2d-1$.
  - For a path with length n, the first step has 2d choices.
  - The ith step has 2d-1 choices (avoid the current position).
  - So $\sigma(n) \le 2d\,(2d-1)^{n-1}$.

Lower bound on $p_c(2)$

- Prove $p_c(2)>0$. In fact we prove $p_c(2) \geq 1/\lambda(d)$.

- We show that when p is sufficiently small, all the clusters are finite, I.e., $\theta(p)=0$.
- $\sigma(n)$: the number of paths starting from origin with length n.
- $N(n)$: the number of length-n paths that appear.

- Look at a particular path, it appears with probability p^n.
- The expectation of $N(n)$ is $E(N(n)) = p^n \sigma(n)$.
- If there is an infinite size cluster, then there exists paths of length n for all n starting from the origin.

Lower bound on $p_c(2)$

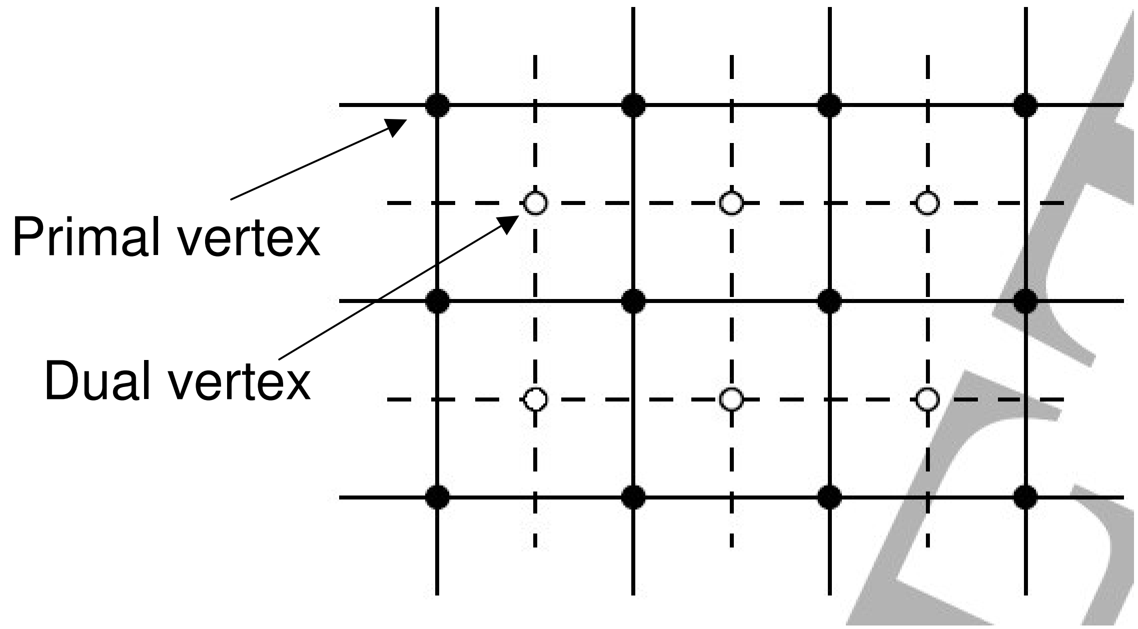
- The expectation of N(n) is $E(N(n)) = p^n \sigma(n)$.
- If there is an infinite size cluster, then there exists paths of length n for all n starting from the origin.

- $\theta(p) \leq$ Prob { N(n) $\geq$ 1 for all n } $\leq E(N(n)) = p^n \sigma(n)$.
- Remember that $\sigma(n)=(\lambda(d)+o(1))^n$ as n goes to infinity.

- $\theta(p) \leq (p\lambda(d) + o(1))^n$.
- Thus $\theta(p) = 0$ if $p\lambda(d)<1$, I.e., $p <1/\lambda(d)$.

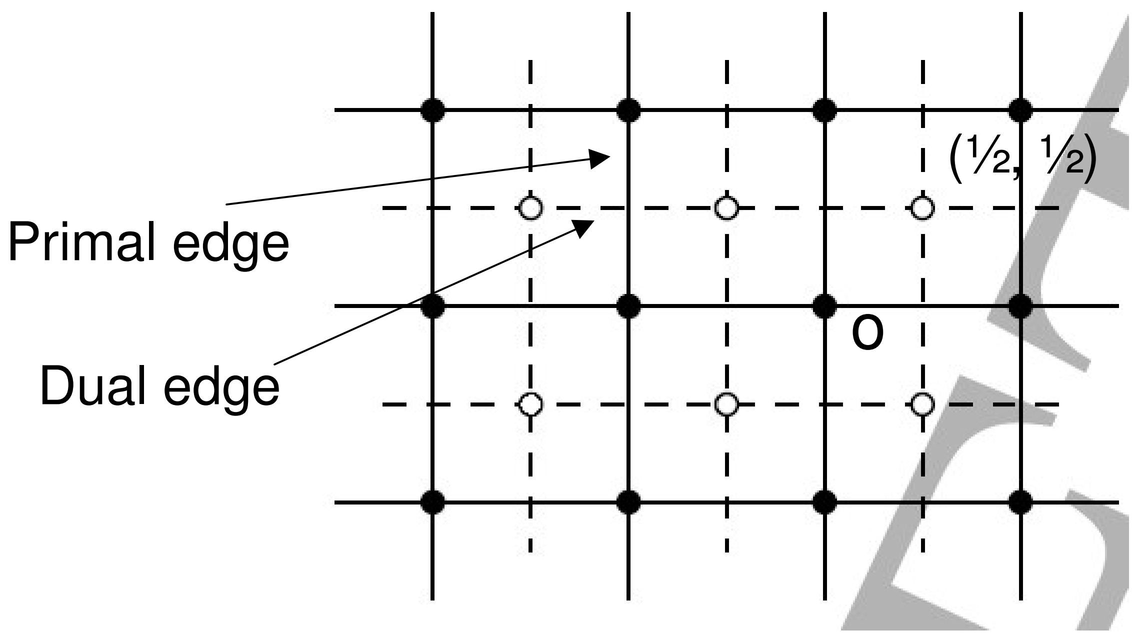
Upper bound on $p_c(2)$

- Prove $p_c(2)<1$.
- We show that $\theta(p)=1$ when p is sufficiently close to 1.
- We use planar duality of a graph.
- For a planar graph (e.g., the grid), map faces to vertices and vertices to faces. The dual of an infinite grid is also a grid.



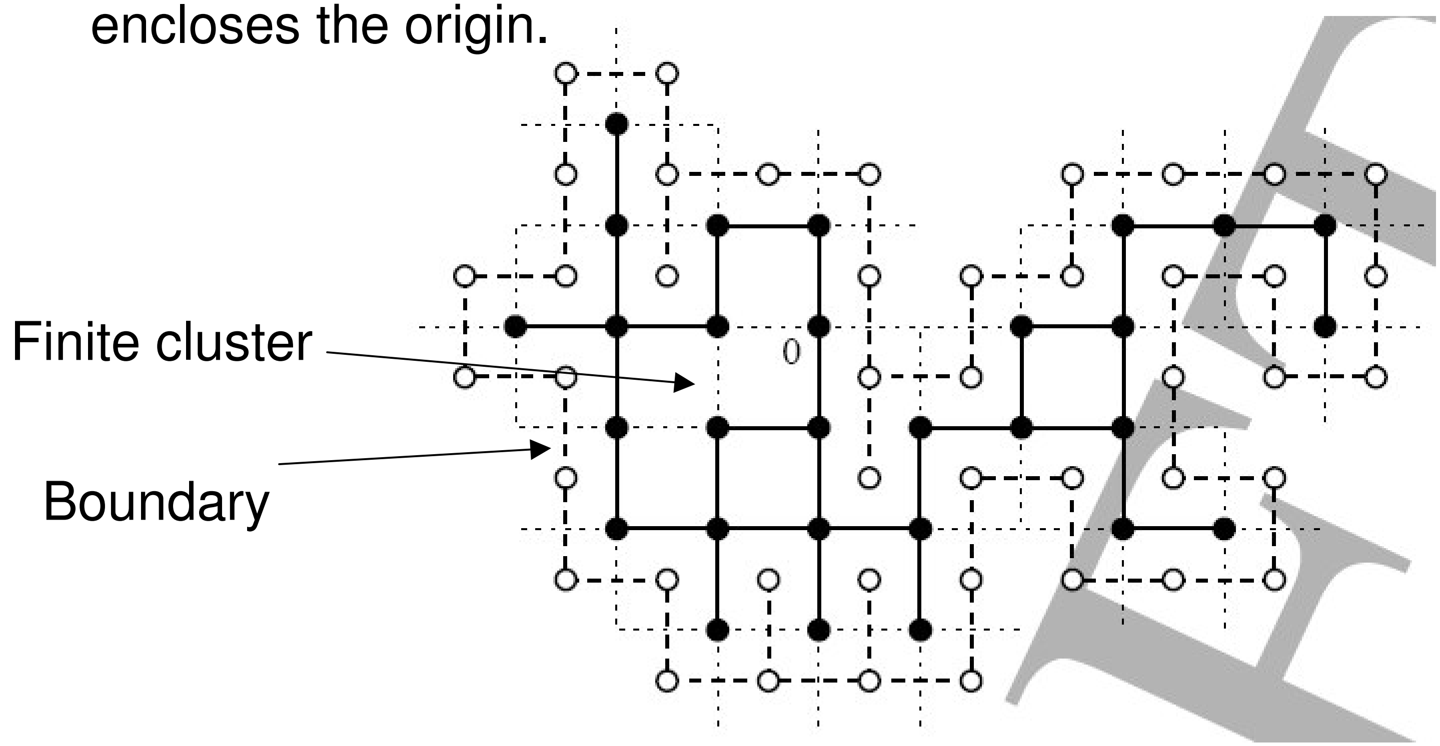
Upper bound on $p_c(2)$

- There is a 1-1 mapping of a primal edge with a dual edge.
- Self-duality: If a primal edge appears (is open), then the dual edge appears (is open).
- The dual lattice $\{x+(½, ½): x \in Z^2\}$.



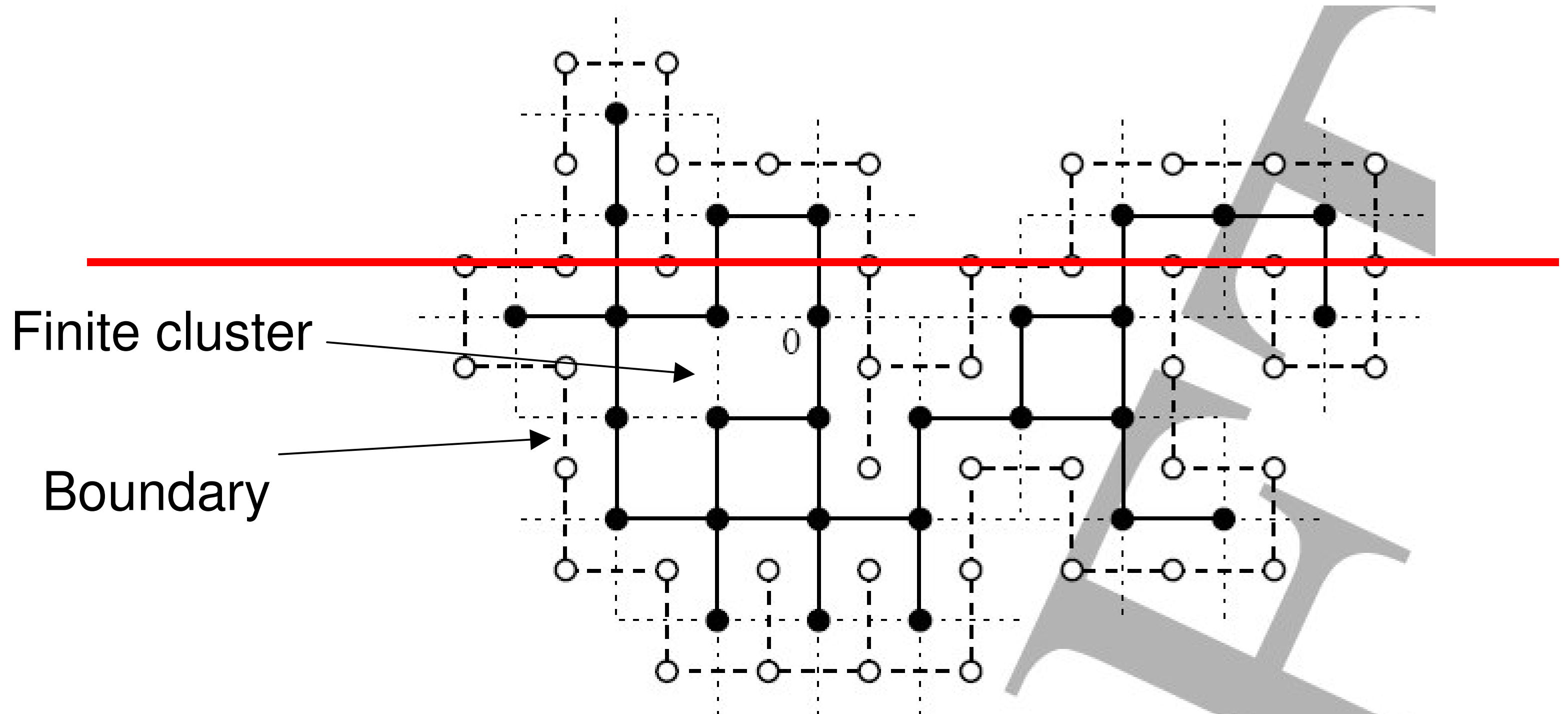
Upper bound on $p_c(2)$

- Suppose the origin is in a finite cluster. Then it is surrounded by a cycle in the dual graph that prevents the origin to reach the infinity.
- Now we count the number of closed circuits in the dual that encloses the origin.



Upper bound on $p_c(2)$

- $\rho(n)$: the number of length-n closed circuits in the dual that encloses the origin.
- Each circuit γ passes through a point $(k+½, ½)$, $0 \le k < n$.
- Thus this circuit contains a self-avoiding walk of length n-1 starting from a vertex $(k+½, ½)$ for some $0 \le k < n$.



Upper bound on $p_c(2)$

- $\rho(n)$: the number of length-n closed circuits in the dual that encloses the origin.
- $\rho(n) \leq n\sigma(n-1)$, where $\sigma(n-1)$ is the # paths of length n-1.
- Thus the total number of such closed circuits, M(n), having lengt

$$\begin{aligned}\sum_{\gamma} P_p(\gamma \text{ is closed}) &\leq \sum_{n=1}^{\infty} q^n n\sigma(n-1) \\ &= \sum_{n=1}^{\infty} qn\{q\lambda(2) + o(1)\}^{n-1} \\ &< \infty\end{aligned}$$

- Where q=1-p, we choose $q\lambda(d)<1$.

$$\sum_{\gamma} P_p(\gamma \text{ is closed}) \to 0 \qquad \text{as } q = 1 - p \downarrow 0,$$

Upper bound on $p_c(2)$

- We find $0<\pi<1$ such that

$$\sum_{\gamma} P_p(\gamma \text{ is closed}) \leq \tfrac{1}{2} \qquad \text{if } p > \pi.$$

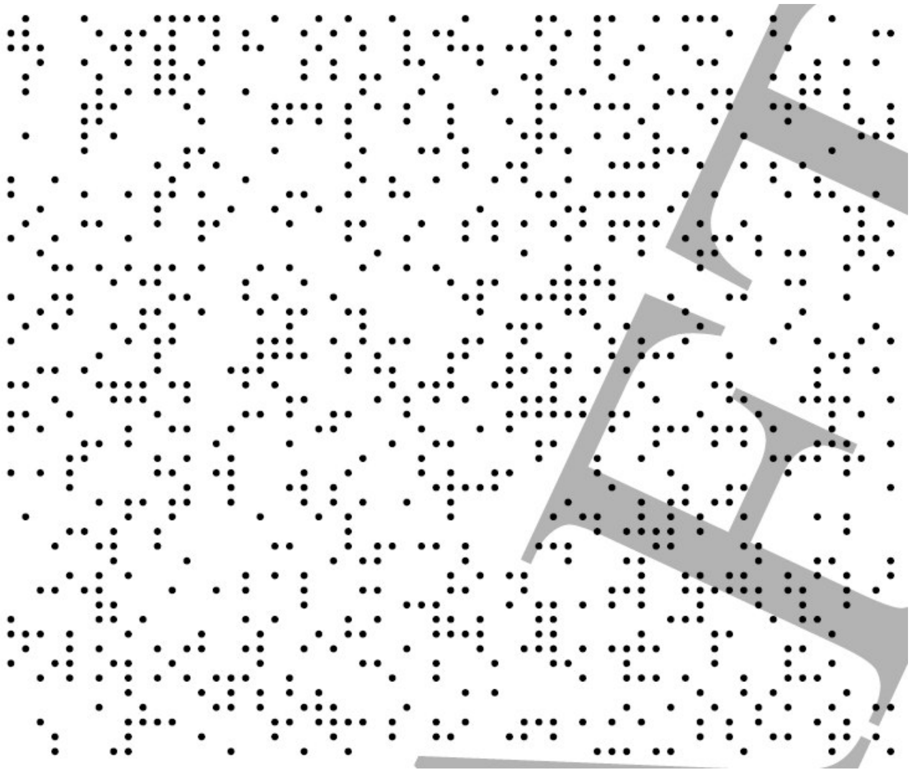
- Thus

$$\begin{aligned} P_p(|C| = \infty) &= P_p\big(M(n) = 0 \text{ for all } n\big) \\ &= 1 - P_p\big(M(n) \geq 1 \text{ for some } n\big) \\ &\geq 1 - \sum_{\gamma} P_p(\gamma \text{ is closed}) \\ &\geq \tfrac{1}{2} \qquad \text{if } p > \pi, \end{aligned}$$

- This proves $p(2)< \pi<1$.

Site Percolation

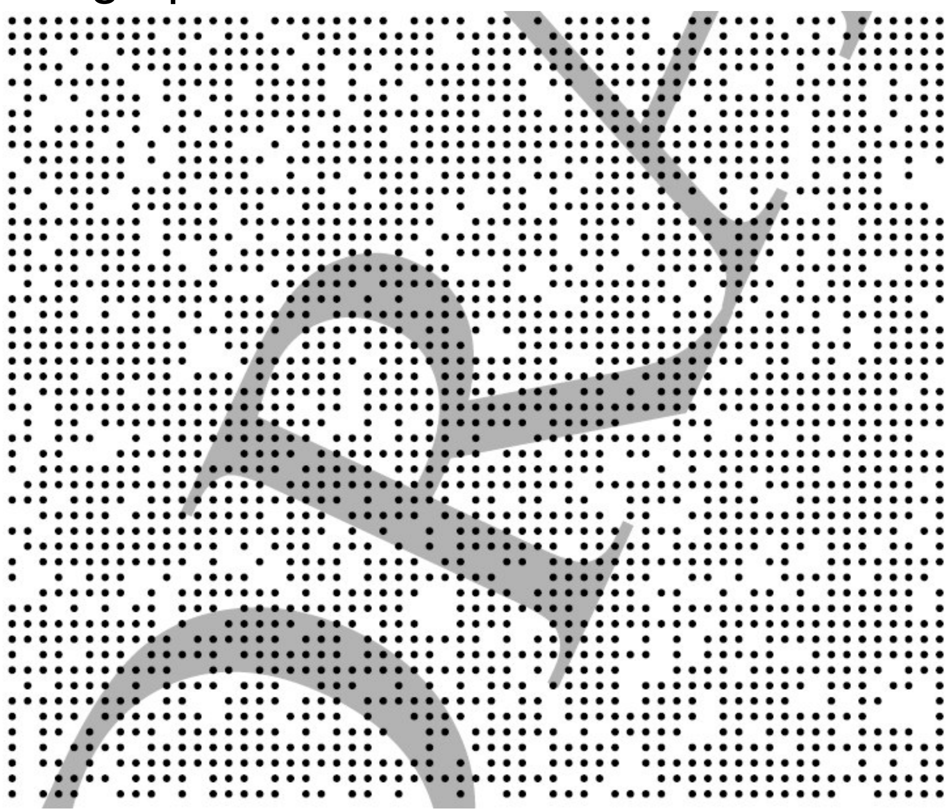
- An infinite grid Z^2, with each vertex to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.3

Site Percolation

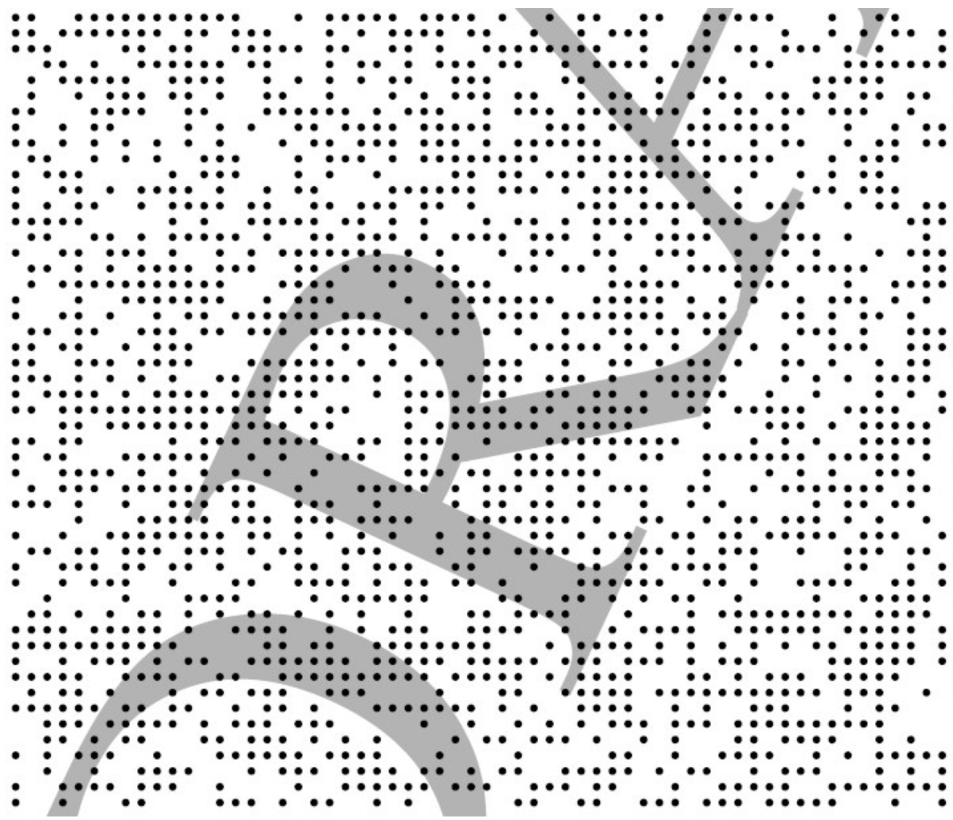
- An infinite grid Z^2, with each vertex to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.80

Site Percolation

- Percolation threshold is still unknown. Simulation shows it's around 0.59. (note this is larger than bond percolation)



p=0.58

Site Percolation

- Site percolation is a generalization of bond percolation.
- Every bond percolation can be represented by a site percolation, but not the other way around.

- Percolation in an infinite connected graph G(V, E).
- Bond percolation: each edge appears with probability p.
- Site percolation: each vertex appears with probability p.

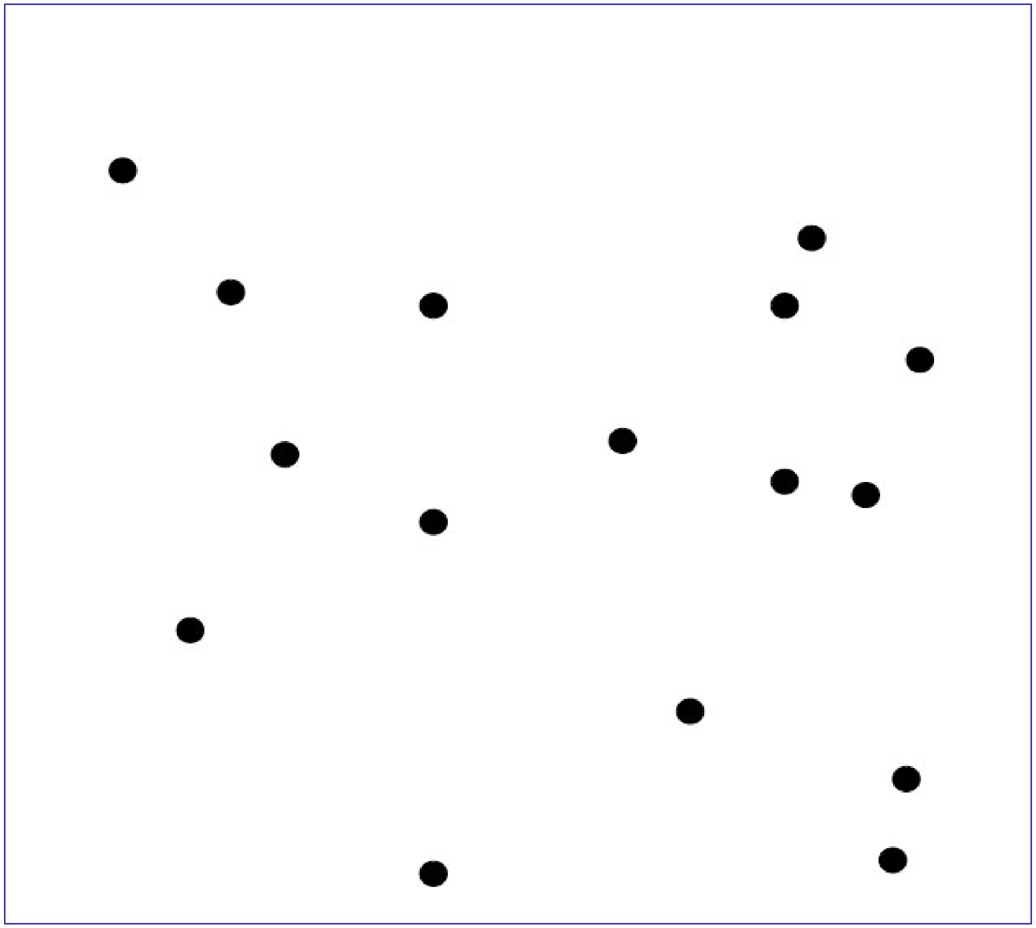
- Denote an arbitrary node as origin, study the cluster containing the origin.

- The percolation threshold of site percolation is always larger than bond percolation.

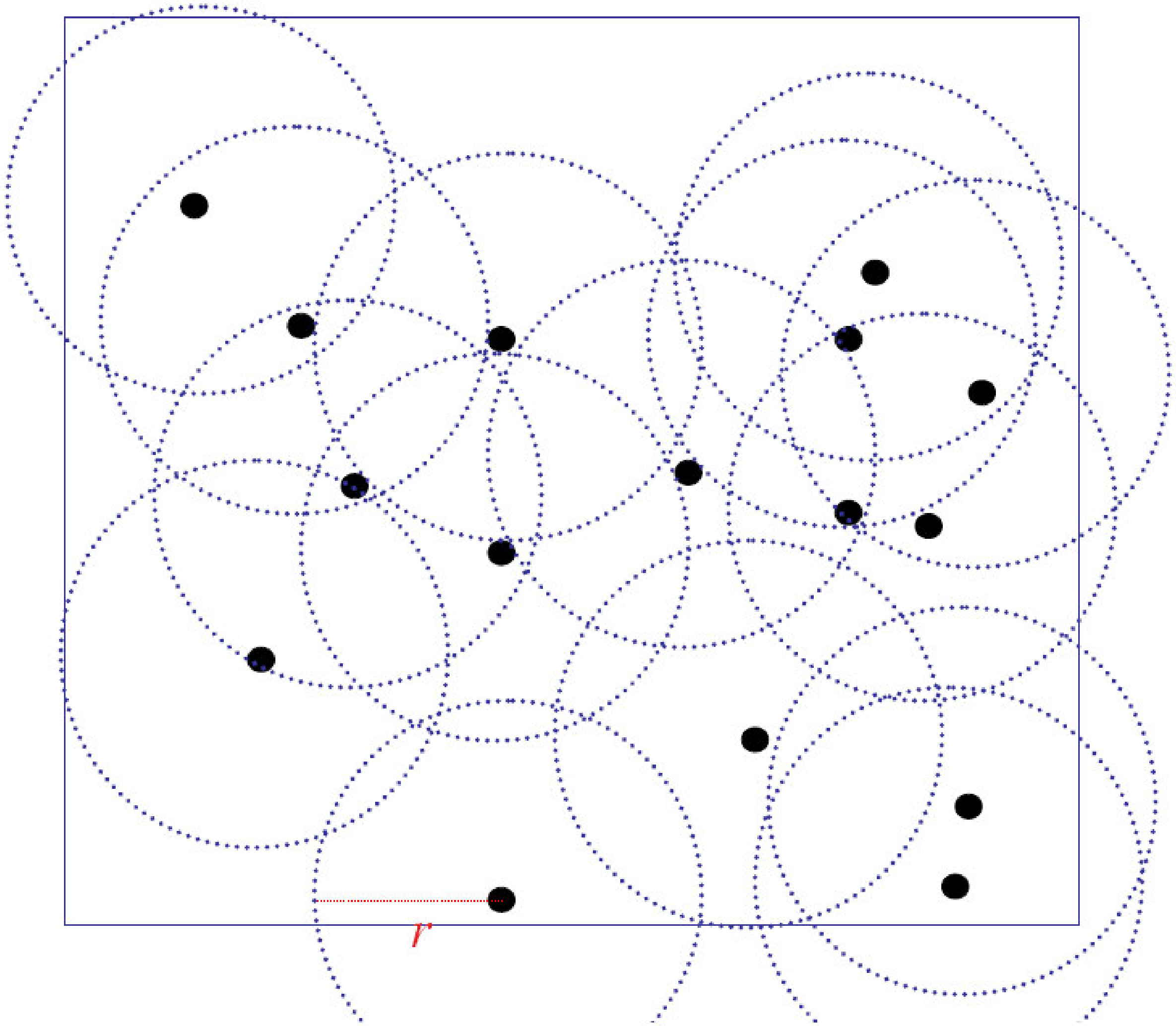
Continuum Percolation

- **Random plane network**, by Gilbert, in J. SIAM 1961.
- Pick points from the plane by a Poisson process with density λ points per unit area.
- Join each pair of points if they are at distance less than r.
- Equivalently,
- In the unit square [0, 1] by [0, 1], throw n points uniformly randomly.
- Connect two nodes with distance less than r.
- This graph is denoted as G(n, r).

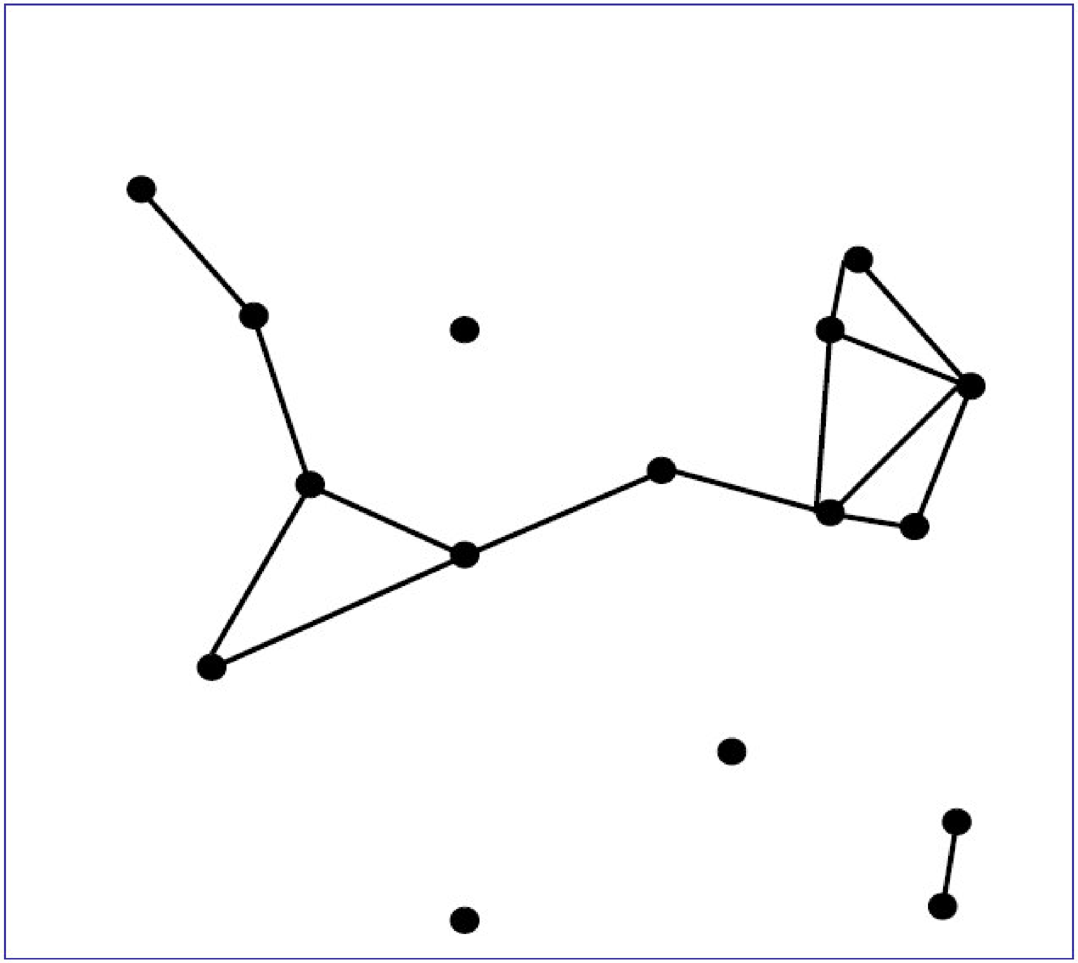
Random geometric graph



Random geometric graph



Random geometric graph



Random geometric graph

- Percolation behavior:
- Given G(n, r), and a desired property (e.g., connectivity), we want to find the smallest radius $r_Q(n)$ such that Q holds with high probability.

- Gupta and Kumar proved:
- Connectivity: if $\pi rn^2 = (\log n + c_n)/n$.
- As c_n goes to infinity, the graph is almost surely connected.
- As c_n goes to –infinity, the graph is almost surely disconnected.

Random geometric graph v.s. random graph

- Erdos-Renyi model of random graphs (Bernoulli random graphs): each pair of vertices is connected by an edge with probability p.

- Random geometric graph: the probability is dependent on the distance.

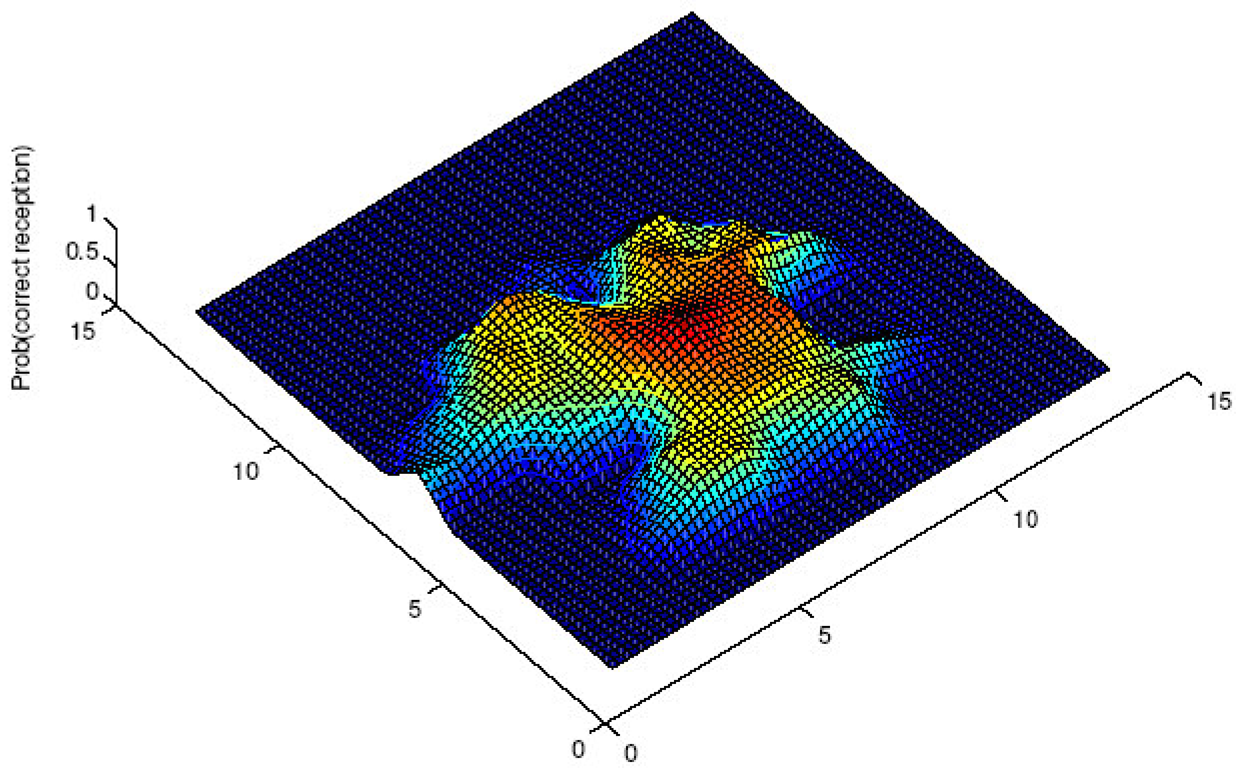
- One of the main question in random graph theory is to determine when a given property is likely to appear.
  - Connectivity.
  - Chromatic number.
  - Matching.
  - Hamiltonian cycle, etc.

Random geometric graph v.s. random graph

- Erdos-Renyi model of random graphs (Bernoulli random graphs): each pair of vertices is connected by an edge with probability p.
- Friedgut and Kalai in 1996 proved that all monotone graph properties have a sharp threshold in Bernoulli random graphs.
- Monotone graph property P: more edges do not hurt the property.
- This is also true in random geometric graphs. Proved by Ashish Goel, Sanatan Rai and Bhaskar Krishnamachari, in STOC 2004.

Percolation in the real world?

- Communication range is not a perfect disk.



Percolation with noisy links

- Each pair of nodes is connected according to some (probabilistic) function of their (random) positions.
- A pair of points (i, j) is connected with probability $g(x_i - x_j)$, where g is a general function that depends only on the distance.
- In order to keep the average degree the same, fix the effective area $e(g) = \int_{x \in \mathbb{R}^2} g(x) dx$
- The average degree = λ e(g).

Percolation with noisy links

- Percolation threshold

$$0 < \lambda_c(g) = \inf\{\lambda : \exists \text{ infinite connected component a.s.}\} < \infty.$$

- Question: what is the relationship between the percolation threshold and the function g?

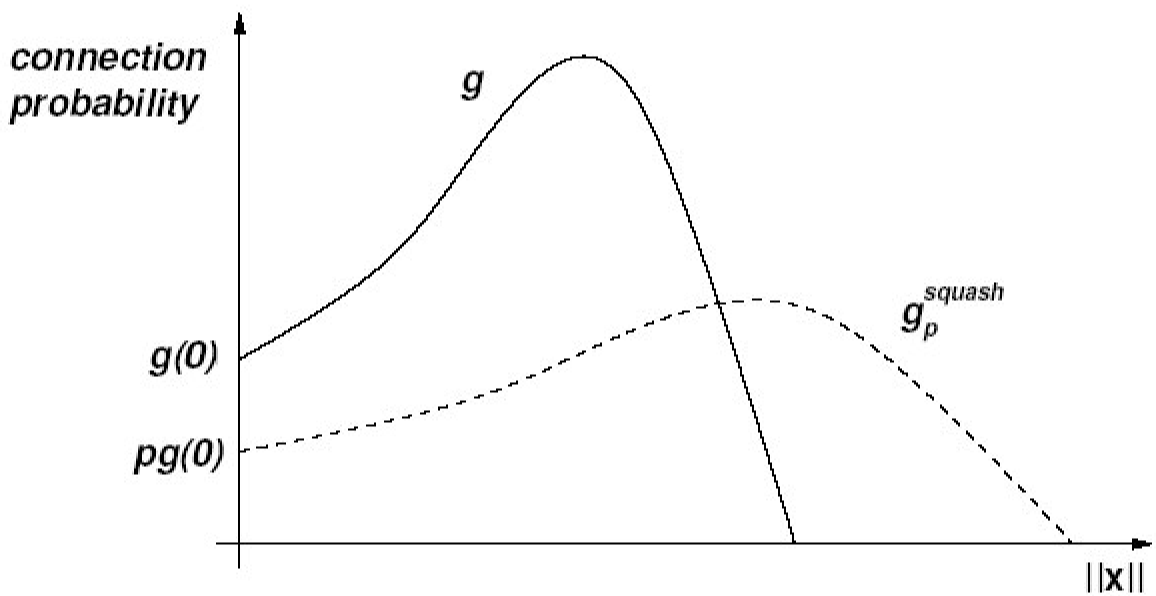
Percolation with noisy links

- Question: what is the relationship between the percolation threshold and the function g?
- Each node is connected to the same number of edges on average. So whom should the node be connected to, in order to have a small percolation threshold?
- Which distribution has the best graph connectivity?
- Should I use reliable short links? Or unreliable long links? Or something more complex, say an annulus?

Squashing

- Probabilities are reduced by a factor of p, but the function is spatially stretched to maintain the same effective area (e.g., the same average degree).

$$g_p^{squash}(x) = p \cdot g(\sqrt{p}x).$$



Squashing

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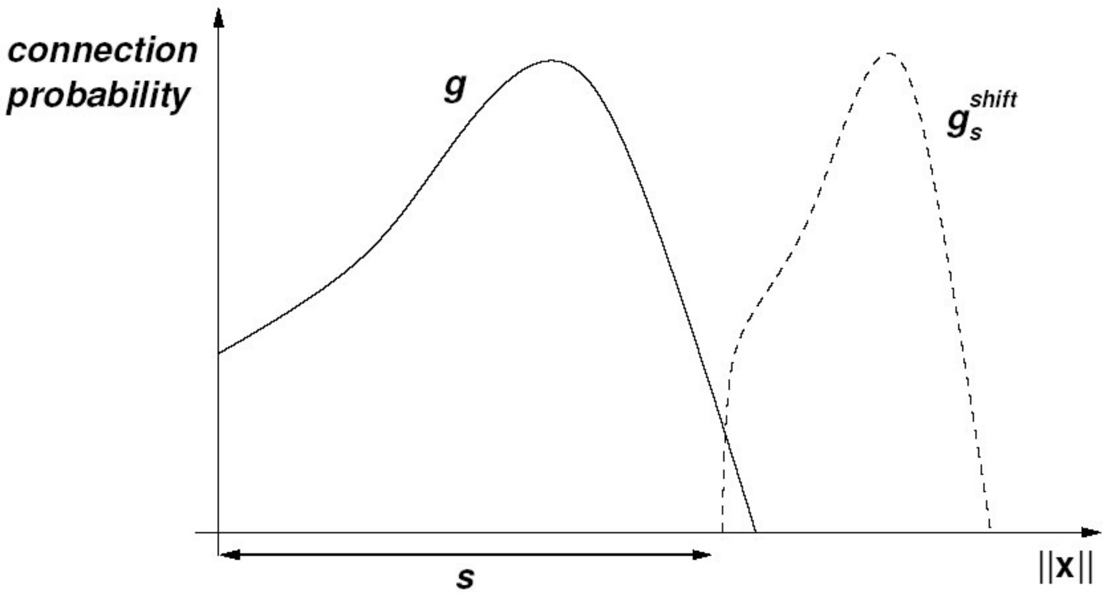
$$g_p^{squash}(x) = p \cdot g(\sqrt{p}x).$$

- Theorem: $$\lambda_c(g) \geq \lambda_c(g_p^{squash}).$$
- It's beneficial for the connectivity to use long unreliable links!
- If the effective area is spread out, then the threshold density goes to 1.
- Question: what makes the difference? The guess is the existence of long links.

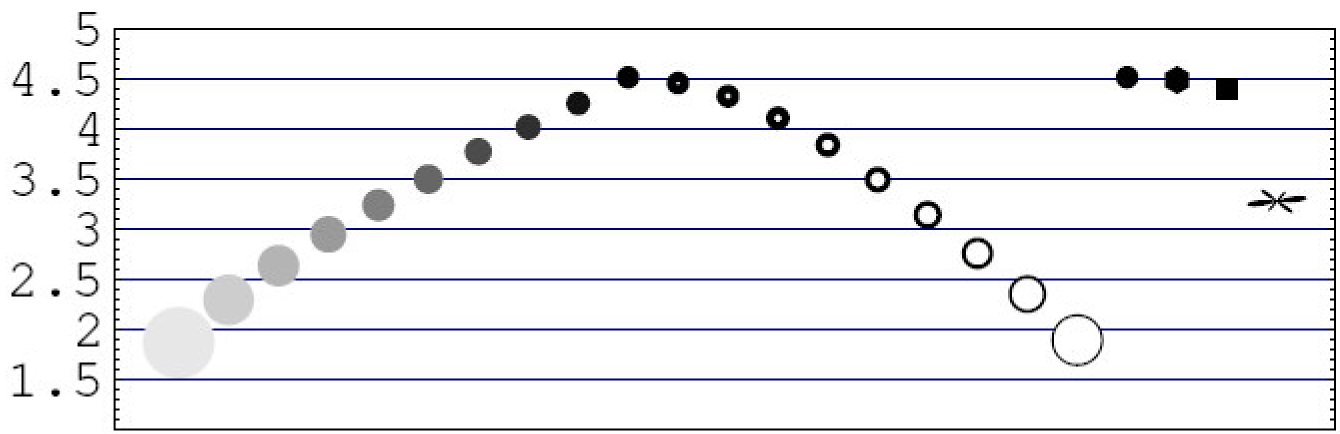
Shifting and squeezing

- Shift the function g outward by a distance s, but squeeze the function after that, so that it has the same effective area.
- Goal: use long links.



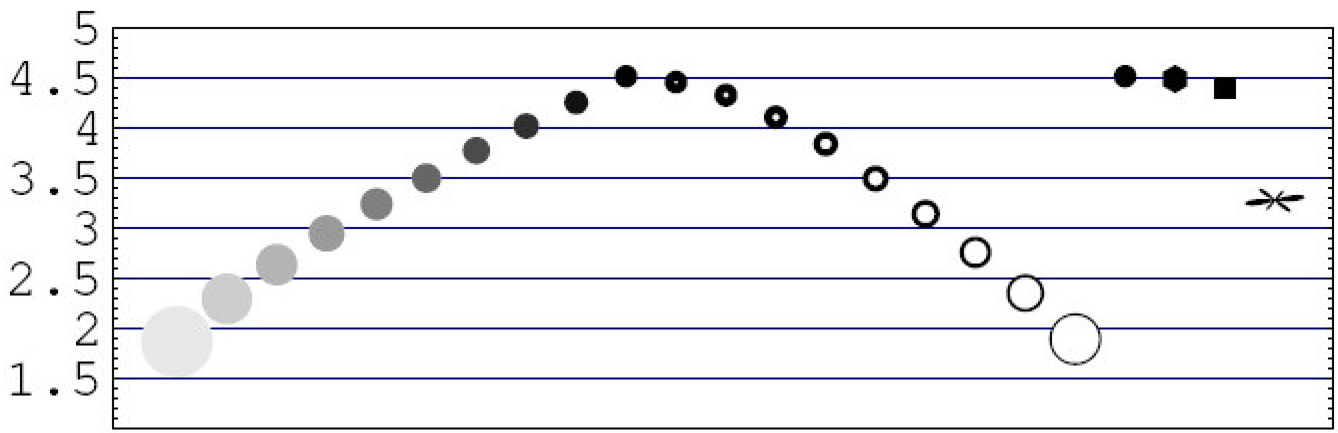
Shifting and squeezing

- Yes it helps percolation! The density threshold goes down.



Connections to points in an annulus

- Points are distributed in the plane by a Poisson process with density λ. Each node is connected to all the nodes inside an annulus A(r) with inner radius r and area 1.
- Theorem: for any critical density λ, one can find a r such that any density above the threshold percolates.



Connection to small-world models

- Kleinberg's model, preferential attachment, etc.
- For grid points, connect two nodes I, j with probability c/d(I, j), where c is a normalization factor.
- Study the property of this network.

Final project

- The final project report is due Dec 22.
- You are welcome to drop by my office for discussions and ideas.